SECTION A

Q1. (a) Prove that a subgroup of a cyclic group is cyclic. Let G be a cyclic group with generator a. If the order of G is infinite, then prove that G is isomorphic to $(\mathbb{Z}, +)$. 8

(b) Find the relative extrema of the function

$f(x, y) = 4y^3 + x^2 - 12y^2 - 36y + 2.$ 8

(c) Prove that in the interval $0 < x < 1$, the function $f(x) = x^2$ is uniformly continuous while $f(x) = \frac{1}{x}$ is not uniformly continuous. 8

(d) Prove that $x_1 = 2,\ x_2 = 1,\ x_3 = 0$ is a feasible solution to the following set of equations :

$$2x_1 - x_2 + 3x_3 = 3$$

$$-6x_1 + 3x_2 + 7x_3 = -9$$

Is the solution basic ? Justify your answer. If the solution is not basic, reduce it to a basic feasible one. 8

(e) Find a bilinear transformation which maps the points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively. 8

Q2. (a) (i) Prove that every group is isomorphic to a group of permutations. 5

(ii) Let $A = \{1, 2, 3\}$ and let S_3 denote the symmetric group on 3 elements. Then is S_3 an abelian or non-abelian group ? 5

(b) (i) Find the volume of the region above the xy-plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$. 7

(ii) Prove that $\lim\limits_{M \to \infty} \int_0^M \frac{dx}{x^4 + 4} = \frac{\pi}{8}$. 8

(c) (i) Let $f(z) = \ln(1 + z)$. Expand $f(z)$ in a Taylor series about $z = 0$. Determine the region of convergence of the series. *8*

(ii) Find Laurent series about the indicated singularity for the function,

$$\frac{e^z}{(z-1)^2};\ z = 1.$$ *7*

Q3. (a) (i) Prove that if $u_n(x)$, $n = 1, 2, 3, \ldots\ldots$ are continuous in $[a, b]$ and if $\Sigma\, u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then $S(x)$ is continuous in $[a, b]$. *5*

(ii) Prove that an absolutely convergent series is convergent. Show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$ is conditionally convergent. *5*

(b) (i) If N is a normal subgroup of a group G and if H is any subgroup of G, then prove that

$$H \vee N = HN = NH$$

where $H \vee N$ denotes the join of H and N. *8*

(ii) State the Second Isomorphism Theorem of groups and apply it to the case $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, $H = \mathbb{Z} \times \mathbb{Z} \times \{0\}$ and $N = \{0\} \times \mathbb{Z} \times \mathbb{Z}$. *7*

(c) Consider the LPP :

Minimize

$$z = 10x_1 + 2x_2$$

subject to

$$x_1 + 2x_2 + 2x_3 \geq 1$$

$$x_1 - 2x_3 \geq -1$$

$$x_1 - x_2 + 3x_3 \geq 3,$$

$$x_i \geq 0, \text{ for } i = 1, 2, 3.$$

Solve the dual of the above LPP and find the minimum value of z. *15*

Q4. (a) (i) State and prove Cauchy's integral formula. Thus evaluate

$$\oint_C \frac{\cos z}{z - \pi}\, dz,$$

where C is the circle $|z - 1| = 3$. 8

(ii) State the Residue Theorem and apply it to evaluate

$$\oint_C \frac{e^z\, dz}{(z-1)(z+3)^2}$$

where C is given by $|z| = \frac{3}{2}$. 7

(b) Prove that the integral domain $\mathbb{Z}$ is a Unique Factorization Domain and a Euclidean Domain. 10

(c) Five workers perform five jobs and the operating cost is given below, but there is a restriction that the worker C cannot perform the third job and B cannot perform the fifth job. Find the optimal assignment and the optimal assignment cost. 15

	I	II	III	IV	V
A	24	29	18	32	19
B	17	26	34	22	–
C	27	16	–	17	25
D	22	18	28	30	24
E	28	16	31	24	27

SECTION B

Q5. (a) Consider a particle of mass m moving in a plane under attractive force $\frac{k}{r^2}$ directed towards the origin, where $k > 0$. Using the polar coordinates (r, θ) write the corresponding Lagrangian and obtain the equations of motion. Also show that the angular momentum is conserved. *8*

(b) A function f, defined on [0, 1], is such that $f(0) = 0$, $f\left(\frac{1}{2}\right) = -1$, $f(1) = 0$. Find the quadratic polynomial p(x) which agrees with f(x) for $x = 0, \frac{1}{2}, 1$.

If $\left|\frac{d^3f}{dx^3}\right| \leq 1$ for $0 \leq x \leq 1$, show that $|f(x) - p(x)| \leq \frac{1}{12}$ for $0 \leq x \leq 1$. *8*

(c) Draw the logic circuit which realises the Boolean function $L = (A + B) . (A + C) + C (A + B . C)$ and simplify it. Draw the simplified circuit also. *8*

(d) In a 2-dimensional flow there are sources at $(a, 0)$, $(-a, 0)$ and sinks at $(0, a)$, $(0, -a)$, all are of equal strength. Determine the stream function and show that the circle through these four points is a streamline. *8*

(e) Solve

$$u_{xx} + \frac{10}{3} u_{xy} + u_{yy} = -\sin(x + y)$$ *8*

Q6. (a) Find the solution of

$$u_x - uu_y + u = 0$$

for the initial values $x_0(s) = 0$, $y_0(s) = s$, $u_0(s) = -2s$.

Does the solution break down for any finite x ? Is the solution unique ? *15*

(b) Find a root of the equation $\sin x + \cos x = 1$, lying in (0, 2), by Regula-Falsi method, correct up to four significant digits. *10*

(c) For a dynamical system having two degrees of freedom, the Lagrangian is given by $L = \frac{m}{2}\left(a^2 \dot{q}_1^2 + \dot{q}_2^2\right) - \frac{k}{2}\left(a^2 + q_2^2\right)$, where q_1 and q_2 are generalized coordinates.

Find the corresponding Hamiltonian and derive the Hamiltonian equations of motion.

Show further that the generalized momentum corresponding to q_1 is constant.

Show that the system exhibits a simple harmonic motion with respect to the generalized coordinate q_2. *15*

Q7. (a) Solve : *10*

$$u_{tt} - u_{xx} = 0, \; 0 < x < z, \; t > 0$$
$$u(0, t) = u(2, t) = 0,$$
$$u(x, 0) = \sin^3 \frac{\pi x}{2},$$
$$u_t(x, 0) = 0.$$

(b) Write down the flow-chart of Runge-Kutta method of 4^{th} order to find $y(0{\cdot}8)$ for $\frac{dy}{dx} = xy$, $y(0) = 2$, taking $h = 0{\cdot}2$.

Also solve the above IVP to find $y(0{\cdot}4)$ by Runge-Kutta method (4^{th} order). *15*

(c) Consider 2-dimensional Navier-Stokes equations of a steady fluid flow.

Show that there exists a stream function $\Psi(x, y)$ for such a flow.

Find the equation satisfied by $\Psi(x, y)$. *1*

Q8. (a) Show that

$$f(x, y, z, p, q) = x^2p^2 + y^2q^2 - 4 = 0$$

$$\text{and } g(x, y, z, p, q) = qy - a = 0,$$

where a is a constant, are compatible and hence solve $f(x, y, z, p, q) = 0$. Is it complete integral ? *15*

(b) State the sufficient condition for convergence of the Gauss-Seidel iteration method and solve the following system of equations by using this method : *15*

$$6{\cdot}7x_1 + 1{\cdot}1x_2 + 2{\cdot}2x_3 = 20{\cdot}5$$

$$2{\cdot}1x_1 - 1{\cdot}5x_2 + 8{\cdot}4x_3 = 28{\cdot}8$$

$$3{\cdot}1x_1 + 9{\cdot}4x_2 - 1{\cdot}5x_3 = 22{\cdot}9$$

(correct up to 3-significant digits)

(c) There is a doublet at (c, 0) in a 2-dimensional flow. A cylinder of radius a ($a < c$) with z-axis as axis of the cylinder was introduced into the flow. Find the complex potential and image system for the flow. *10*