SECTION A

Q1. (a) Let V be a vector space of the dimension n over a field F. Then show that V is isomorphic to F^n. *8*

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by

$$T(x, y, z) = (x, z, -2y - z) \text{ and let } f(u) = -u^3 + 2.$$

Then find f(T). *8*

(c) Test the convergence of improper integral *8*

$$\int_a^b \frac{dx}{(x-a)^n}.$$

(d) If $u = z \sin\left(\frac{y}{x}\right)$; where $x = 3r^2 + 3s$, $y = 4r - 2s^3$, $z = 2r^2 - 3s^2$; then find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$. *8*

(e) If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, represents two intersecting straight lines, then show that the square of the distance of the point of intersection from the origin is $\frac{c(a+b) - (f^2 + g^2)}{ab - h^2}$. *8*

Q2. (a) If $S_1 = \{(x, y, z) \mid x + 2y + z = 0\}$ and $S_2 = \{(x, y, z) \mid x + y - z = 0\}$ are subspaces of $\mathbb{R}^3$, then

(i) find a basis of $S_1 \cap S_2$.

(ii) determine $\dim (S_1 + S_2)$.

(iii) describe $S_1 \cap S_2$ and $S_1 + S_2$ geometrically. *5+3+2=10*

(b) Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

Then show that

(i) $D_1 f(0, 0)$ and $D_2 f(0, 0)$ exist.

(ii) $f(x, y)$ is continuous at $(0, 0)$ by ε-δ method.

(iii) $f(x, y)$ is not differentiable at $(0, 0)$.

where D_1 and D_2 denote partial derivatives with respect to x and y respectively. 4+4+7=15

(e) (i) Find the equation of the plane which passes through the point $(2, 1, -1)$ and is orthogonal to each of the planes $x - y + z = 1$, $3x + 4y - 2z = 0$. 8

(ii) Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$ is a great circle. 7

Q3. (a) Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$, using Lagrange's Multiplier method. 10

(b) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. Prove that T is invertible and find T^{-1}. 15

(c) (i) Find the coordinates of the vertex, focus and the length of the latus rectum of the principal sections of the paraboloid given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$. 8

(ii) Find the nature of the quadric surface given by the equation $2x^2 + 5y^2 + 3z^2 - 4x + 20y - 6z = 5$. Also find its associated characteristics, principal axes and principal planes. 7

Q4. (a) Prove that the straight lines

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} \text{ and } \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$$

intersect and find the equation of the plane containing them. Also find their point of intersection. 10

(b) Show that the matrix

$$A = \begin{bmatrix} 7 & -6 & 6 \\ 2 & 0 & 4 \\ 1 & -2 & 6 \end{bmatrix}$$

is diagonalizable and find a spectral decomposition of the matrix A. 15

(c) Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and planes $y + z = 4$ and $z = 0$. 15

SECTION B

Q5. (a) Find the solution of the differential equation

$$2x^2y\left(\frac{dy}{dx}\right) = \tan(x^2y^2) - 2xy^2.$$

What will be the definite value of the arbitrary constant, appearing in the solution, on coordinate axes ? 8

(b) Identify that one solution of the equation

$$xy'' + (x-1)y' - y = 0$$

which is of the form $ce^{\pm ax}$ and then find the other solution by method of reduction of order. 8

(c) Four bars are joined together to form a rhombus. The bars are uniform and each bar is of weight W. A rhombus is suspended vertically from one of the joints and a spherical ball of weight S is balanced inside the rhombus so as to keep the system intact.

If 2θ is the angle at a fixed joint in the state of equilibrium, then find the ratio of weight of the rhombus to that of the spherical ball in terms of the radius of the sphere, the length of a bar and the angle 'θ'. 8

(d) In a central orbit, the central force is given as $\mu u^3(3 + 2a^2u^2)$. If a particle is projected at a distance a with velocity $\sqrt{\frac{5\mu}{a^2}}$ in a direction making an angle $\tan^{-1}\frac{1}{2}$ with the radius, then show that equation of its path can be written as $r = a\tan\theta$. 8

(e) If $\nabla \cdot \vec{E} = 0$, $\nabla \cdot \vec{H} = 0$, $\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t}$ and $\nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t}$, then show that $\nabla^2 \vec{H} = \frac{\partial^2 \vec{H}}{\partial t^2}$ and $\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$. 8

Q6. (a) Solve the differential equation

$$y\,(xy + 2x^2y^2)\,dx + x\,(xy - x^2y^2)\,dy = 0.$$

Verify that the obtained solution satisfies the given differential equation. *10*

(b) A particle slides down the smooth curve, $y = a \sinh\left(\frac{x}{a}\right)$, the axis of x being horizontal and the axis of y downwards, starting from rest at the point where the tangent is inclined at α to the horizon.

Show that the particle will leave the curve when it has fallen through a vertical distance $a \sec \alpha$. *15*

(c) (i) Show that the area bounded by a simple closed C is given by $\frac{1}{2}\oint_C (x\,dy - y\,dx)$. Hence obtain the area of an ellipse. *8*

(ii) Evaluate $\oint_\Gamma (e^x\,dx + 2y\,dy - dz)$ by using Stokes' theorem, where Γ is the curve $x^2 + y^2 = a^2$, $z = h$. *7*

Q7. (a) A uniform ladder of 10 m length and of 10 kg weight rests with its foot on the rough ground and its upper end against a smooth wall, the inclination to the vertical being α. A force P is applied horizontally to the ladder at a point distant 3 m from the foot, so as to make the foot approach the wall.

Prove that the force P must exceed $\frac{100}{7}\left(\mu + \frac{1}{2}\tan\alpha\right)$, where μ is the coefficient of friction at the foot. *10*

(b) Find the complete solution of

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{\log x \sin(\log x) + 1}{x}.$$

15

(e) (i) Given that $u = x + y + z$, $v = x^2 + y^2 + z^2$ and $w = xy + yz + zx$. Show that the vectors grad u, grad v and grad w are coplanar. *8*

(ii) For the curve given by $\vec{r} = \left(2t, t^2, \frac{t^3}{3}\right)$, find the curvature and torsion at $t = 1$. *7*

Q8. (a) (i) Given that the vectors $\vec{f}$ and $\vec{g}$ are irrotational. Show that the vector $\vec{f} \times \vec{g}$ is solenoidal.

(ii) Show that the vector $\vec{q} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational and find a scalar function φ such that $\vec{q} = \text{grad}\,\varphi$. *3+7=10*

(b) Find singular solution of the differential equation

$$y^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right)\tan^2\beta + y^2\sec^2\beta - x^2\tan^2\beta = 0$$

directly and from its complete primitive. Determine tac-locus. Show that the envelope of family of curves, which is represented by the given equation, is $y = \pm x \tan\beta$. *15*

(c) An elliptic lamina is completely immersed in water with its plane vertical. Its minor axis is horizontal and is at a depth h. Determine the centre of pressure. *15*