MATHEMATICS

Paper – I

Time Allowed : ***Three Hours***

Maximum Marks : ***200***

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

There are ***EIGHT*** *questions in all, out of which* ***FIVE*** *are to be attempted.*

Questions no. 1 and 5 are **compulsory**. *Out of the remaining* ***SIX*** *questions,* ***THREE*** *are to be attempted selecting at least* ***ONE*** *question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

Answers must be written in ***ENGLISH*** *only.*

SECTION A

Q1. (a) Consider the following quadratic form :

$$q(x, y, z) = 2x^2 + 2y^2 + 6z^2 + 2xy - 6yz - 6zx,$$

where (x, y, z) are the coordinates of the vector X with respect to the standard basis {(1, 0, 0), (0, 1, 0), (0, 0, 1)} of R^3. Find the expression of q(x, y, z) with respect to the basis

$$B = \left\{\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}.$$

Is q positive definite ? Justify your answer. *8*

(b) Prove that the product of two Hermitian matrices A, B is Hermitian if and only if A and B commute. Give an example of a pair of 3 × 3 symmetric matrices such that their product is again symmetric (do not consider only diagonal matrices) and also check whether they commute or not. *8*

(c) Using Beta and Gamma functions, evaluate the following integrals : *4+4*

(i) $\displaystyle\int_0^2 x(8 - x^3)^{1/3}\, dx$

(ii) $\displaystyle\int_0^1 \frac{x^2\, dx}{\sqrt{1 - x^5}}$

(d) Evaluate $\displaystyle\iint_R x^2\, dx\, dy,$

where R is the region in the first quadrant bounded by the hyperbola xy = 16 and the lines y = x, y = 0 and x = 8. *8*

(e) Find the equation of the plane passing through the points (1, – 1, 1) and (– 2, 1, – 1) and perpendicular to the plane 2x + y + z + 5 = 0. *8*

Q2. (a) Express the polynomial $f(x) = x^2 + 4x - 3$ over R as linear combination of polynomials $e_1 = x^2 - 2x + 5$, $e_2 = 2x^2 - 3x$, $e_3 = x + 3$. Also, show that the set $\{e_1, e_2, e_3\}$ forms a basis of all quadratic polynomials over R. *10*

(b) Find the shortest distance between the line $y = 10 - 2x$ and the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

using Lagrange's method of multipliers. *15*

(c) Find the equation of the cone whose vertex is (1, 2, 1) and which passes through the circle $x^2 + y^2 + z^2 = 5$, $x + y - z = 1$. *15*

Q3. (a) Does $f(x) = x + \frac{1}{x}$ in $\left[\frac{1}{2}, 3\right]$ satisfy the conditions of the mean value theorem ? If yes, then justify your answer and find $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \left(a = \frac{1}{2}, b = 3\right).$$ *10*

(b) Given the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$, find a similarity transformation that diagonalises the matrix A. *15*

(c) Show that the straight lines whose direction cosines are given by the equations $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ (where a, b, c, u, v, w are constants) are parallel if $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$ and perpendicular if $a^2(v + w) + b^2(w + u) + c^2(u + v) = 0$. *15*

Q4. (a) Find the equation of the sphere passing through the points (1, 1, 2), (1, – 1, 2) and having centre on the line $x + y - z - 1 = 0 = 2x + y - z - 2$. *10*

(b) Using the Cayley-Hamilton theorem, find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 2 & 1 \end{bmatrix}$. *15*

(c) Find the whole area included between the curve $x^2 y^2 = a^2 (y^2 - x^2)$ and its asymptotes. *15*

SECTION B

Q5. (a) Solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$$

by the method of variation of parameters. *8*

(b) Solve the differential equation

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right).$$ *8*

(c) A particle is projected in a direction making an angle α with the horizon. It passes through the two points (x_1, y_1) and (x_2, y_2). Prove that

$$\tan\alpha = \frac{y_1 R}{Rx_1 - x_1^2} = \frac{x_2^2 y_1 - x_1^2 y_2}{x_1 x_2 (x_2 - x_1)},$$

where R denotes the horizontal range. *8*

(d) Four light rods are joined smoothly to form a quadrilateral ABCD. Let P and Q be the mid-points of an opposite pair of rods and these points are connected by a string in a state of tension T. Let R and S be the mid-points of the other opposite pair of rods and these points are connected by a light rod in a state of thrust X. Show that

$$T\,.\,(RS) = X\,.\,(PQ).$$ *8*

(e) If $\vec{F} = \left(y\frac{\partial\phi}{\partial z} - z\frac{\partial\phi}{\partial y}\right)\hat{i} + \left(z\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial z}\right)\hat{j} + \left(x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x}\right)\hat{k}$,

then prove that

$$\vec{F} - (\vec{r} \times \nabla\phi) = \vec{F}\,.\,\vec{r} = \vec{F}\,.\,\nabla\phi = 0.$$ *8*

Q6. (a) Solve the differential equation

$$(D^4 + D^2 + 1)y = e^{-x/2} \cos\left(\frac{1}{2} x \sqrt{3}\right).$$ *10*

(b) A particle is moving in a medium with central acceleration P. The medium is a resisting medium in which resistance = kv^2, v being the velocity.

Let s be the arc-length; (r, θ) be plane polar coordinates; $u = \frac{1}{r}$ and M_0 be the initial moment of momentum about the centre of force. Show that the equation of the path of the particle is

$$Pe^{2ks} = M_0^2 u^2 \left(u + \frac{d^2u}{d\theta^2}\right).$$ *15*

(c) Let $\vec{a}$ and $\vec{b}$ be any two vector point functions defined on Euclidean space R^3. Derive the vector identity for $\nabla(\vec{a} \cdot \vec{b})$. Verify that identity for grad(grad ϕ . grad ψ), where $\phi = 3x^2y$, $\psi = xz^2 - 2y$. *15*

Q7. (a) State Gauss' Divergence Theorem completely. Verify the theorem for a field vector $\vec{f} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 9$; $z = 0$, $z = 4$. *10*

(b) Find the general solution of the differential equation

$$(1 + 2x)^2 \frac{d^2y}{dx^2} - 6(1 + 2x) \frac{dy}{dx} + 16y = 8(1 + 2x)^2.$$ *15*

(c) Given a solid in the shape of a double cone bounded by two equal circular ends. The solid floats in a liquid, whose density is twice that of the cone, with its axis horizontal. Prove that the equilibrium is stable or unstable according as the semi-vertical angle is less than or greater than 60°. *15*

Q8. (a) If the mass density at any point of a cord varies as the radius of curvature of the curve in which it hangs freely under gravity, then prove that this curve is the catenary of uniform strength. *10*

(b) (i) Reduce the differential equation $axyp^2 + (x^2 - ay^2 - b)p - xy = 0$, $\left(p = \frac{dy}{dx}\right)$ to Clairaut's form and find the general solution. *8*

(ii) Find the singular solution of the differential equation

$$9p^2(2 - y)^2 = 4(3 - y), \left(p = \frac{dy}{dx}\right).$$ *7*

(c) Prove that :

(i) Principal normals at consecutive points on a curve in a space do not intersect unless its torsion is zero. *7*

(ii) Principal normal of a curve in a space will be binormal of another curve if the curvature of the given curve is proportional to $(k^2 + z^2)$. *8*