

# MATHEMATICS

## Paper – I

Time Allowed : **Three Hours**

Maximum Marks : **200**

### Question Paper Specific Instructions

*Please read each of the following instructions carefully before attempting questions :*

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are **compulsory**. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

*Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.*

*All questions carry equal marks. The number of marks carried by a question/part is indicated against it.*

*Answers must be written in **ENGLISH** only.*

*Unless otherwise mentioned, symbols and notations have their usual standard meanings.*

*Assume suitable data, if necessary, and indicate the same clearly.*

## SECTION A

- Q1. (a) If  $A$  is a skew-symmetric matrix and  $I + A$  be a non-singular matrix, then show that  $(I - A)(I + A)^{-1}$  is orthogonal. 8

- (b) By applying elementary row operations on the matrix

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix},$$

reduce it to a row-reduced echelon matrix. Hence find the rank of  $A$ . 8

- (c) Given that  $f(x + y) = f(x)f(y)$ ,  $f(0) \neq 0$ , for all real  $x, y$  and  $f'(0) = 2$ . Show that for all real  $x$ ,  $f'(x) = 2f(x)$ . Hence find  $f(x)$ . 8

- (d) Find the Taylor's series expansion for the function

$$f(x) = \log(1 + x), \quad -1 < x < \infty,$$

about  $x = 2$  with Lagrange's form of remainder after 3-terms. 8

- (e) If the straight lines, joining the origin to the points of intersection of the curve  $3x^2 - xy + 3y^2 + 2x - 3y + 4 = 0$  and the straight line  $2x + 3y + k = 0$ , are at right angles, then show that  $6k^2 + 5k + 52 = 0$ . 8

Q2. (a) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be defined by  $T(x, y, z) = (2x, -3y, x + y)$ , and  $B_1 = \{(-1, 2, 0), (0, 1, -1), (3, 1, 2)\}$  be a basis of  $\mathbb{R}^3$ . Find the matrix representation of  $T$  relative to the basis  $B_1$ . 10

(b) Using Lagrange's multiplier, show that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube. 15

(c) Prove that the angle between two straight lines whose direction cosines are given by  $l + m + n = 0$  and  $fmn + gnl + hlm = 0$  is  $\frac{\pi}{3}$ , if  $\frac{1}{f} + \frac{1}{g} + \frac{1}{h} = 0$ . 15

Q3. (a) Find the asymptotes of the curve  $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$ . 10

(b) When is a matrix  $A$  said to be similar to another matrix  $B$ ?  
Prove that

(i) if  $A$  is similar to  $B$ , then  $B$  is similar to  $A$ .

(ii) two similar matrices have the same eigenvalues.

Further, by choosing appropriately the matrices  $A$  and  $B$ , show that the converse of (ii) above may not be true. 15

(c) A point  $P$  moves on the plane  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ , which is fixed. The plane through  $P$  and perpendicular to  $OP$  meets the axes in  $A, B, C$  respectively. The planes through  $A, B, C$  parallel to  $yz, zx$  and  $xy$  planes respectively intersect at  $Q$ . Prove that the locus of  $Q$  is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}. \quad 15$$

Q4.

(a)

Let  $P$  be the vertex of the enveloping cone of the ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ . If the section of this cone made by the plane  $z = 0$  is a rectangular hyperbola, then find the locus of  $P$ .

10

(b) (i) Verify Cayley-Hamilton theorem for the matrix  $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ , hence

find its inverse. Also, express  $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$  as a linear polynomial in  $A$ .

(ii) Express the vector  $(1, 2, 5)$  as a linear combination of the vectors  $(1, 1, 1)$ ,  $(2, 1, 2)$  and  $(3, 2, 3)$ , if possible. Justify your answer.  $9+6=15$

(c) (i) Evaluate :

$$\lim_{x \rightarrow 1} (x-1) \tan \frac{\pi x}{2}.$$

(ii) Evaluate the following integral :

$$\int_{-\infty}^{\infty} x e^{-x^2} dx$$

6+9=15

## SECTION B

Q5. (a) Solve the initial value problem :

$$(2x^2 + y) dx + (x^2y - x) dy = 0, \quad y(1) = 2.$$

8

(b) Solve the differential equation

$$\frac{d^2y}{dx^2} - 3 \frac{dy}{dx} - 4y = 16x - 12e^{2x}.$$

8

(c) If the radial and transverse velocities of a particle are proportional to each other, then prove that the path is an equiangular spiral. Further, if radial acceleration is proportional to transverse acceleration, then show that the velocity of the particle varies as some power of the radius vector.

8

(d) A cylinder of radius 'r', whose axis is fixed horizontally, touches a vertical wall along a generating line. A flat beam of length  $l$  and weight 'W' rests with its extremities in contact with the wall and the cylinder, making an angle of  $45^\circ$  with the vertical. Prove that the reaction of the cylinder is  $\frac{W\sqrt{5}}{2}$  and the pressure on the wall is  $\frac{W}{2}$ . Also, prove that the ratio of radius of the cylinder to the length of the beam is  $5 + \sqrt{5} : 4\sqrt{2}$ .

8

(e) Prove that for a vector  $\vec{a}$ ,

$$\nabla (\vec{a} \cdot \vec{r}) = \vec{a}; \text{ where } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \quad r = |\vec{r}|.$$

Is there any restriction on  $\vec{a}$  ?

Further, show that

$$\vec{a} \cdot \nabla \left( \vec{b} \cdot \nabla \frac{1}{r} \right) = \frac{3(\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r})}{r^5} - \frac{\vec{a} \cdot \vec{b}}{r^3}$$

Give an example to verify the above.

8

- Q6.** (a) Find one solution of the differential equation

$$(x^2 + 1) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$$

by inspection and using that solution determine the other linearly independent solution of the given equation. Obtain the general solution of the given differential equation. 10

- (b) A particle of mass 5 units moves in a straight line towards a centre of force and the force varies inversely as the cube of distance. Starting from rest at the point A distant 20 units from centre of force O, it reaches a point B distant 'b' from O. Find the time in reaching from A to B and the velocity at B. When will the particle reach at the centre ? 15

- (c) A tangent is drawn to a given curve at some point of contact. B is a point on the tangent at a distance 5 units from the point of contact. Show that the curvature of the locus of the point B is

$$\frac{[25 \kappa^2 \tau^2 (1 + 25 \kappa^2) + \{\kappa + 5 \frac{d\kappa}{ds} + 25 \kappa^3\}]^{1/2}}{(1 + 25 \kappa^2)^{3/2}}.$$

Find the curvature and torsion of the curve  $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ . 15

- Q7.** (a) Derive intrinsic equation

$$x = c \log (\sec \psi + \tan \psi)$$

of the common catenary, where symbols have usual meanings.

Prove that the length of an endless chain, which will hang over a circular pulley of radius 'a' so as to be in contact with  $\frac{2}{3}$  of the circumference of the pulley, is

$$a \left\{ \frac{4\pi}{3} + \frac{3}{\log (2 + \sqrt{3})} \right\}.$$

10

- (b) Solve the differential equation

$$x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1-x)^2}. \quad 15$$

- (c) Given a portion of a circular disc of radius 7 units and of height 1.5 units such that  $x, y, z \geq 0$ .

Verify Gauss Divergence Theorem for the vector field

$$\vec{f} = (z, x, 3y^2z)$$

over the surface of the above mentioned circular disc. 15

- Q8. (a) Derive expression of  $\nabla f$  in terms of spherical coordinates.

Prove that

$$\nabla^2 (fg) = f \nabla^2 g + 2 \nabla f \cdot \nabla g + g \nabla^2 f$$

for any two vector point functions  $f(r, \theta, \phi)$  and  $g(r, \theta, \phi)$ .

Construct one example in three dimensions to verify this identity. 10

- (b) Reduce the differential equation

$$xp^2 - 2yp + x + 2y = 0, \quad \left( p = \frac{dy}{dx} \right),$$

to Clairaut's form and obtain its complete primitive. Also, determine a singular solution of the given differential equation. 15

- (c) A sphere of radius 'a', and having density half of that of water, is completely immersed at the bottom of a circular cylinder of radius 'b', which is filled with water to depth 'd'. The sphere is set free and takes up its position of equilibrium. Show that the loss of potential energy this way is

$$W \left( d - \frac{11}{8}a - \frac{a^3}{3b^2} \right),$$

where W is the weight of the sphere. 15