IAS
PREVIOUS YEARS QUESTIONS (2020-1983)
SEGMENT-WISE

COMPLEX ANALYSIS

2020

❖ Evaluate the integral $\int_C \left(z^2+3z\right)dz$ counterclockwise from (2, 0) to (0, 2) along the curve C, where C is the circle $|z| = 2$. **[10]**

❖ Using contour integration, evaluate the integral $\int_0^{2\pi} \frac{1}{3+2\sin\theta}d\theta$. **[20]**

❖ If $v(r,\theta)=\left(r-\frac{1}{r}\right)\sin\theta, r\neq 0$, then find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ **[15]**

2019

❖ Suppose f(z) is analytic function on a domain D in ⊄ and satisfies the equation $\text{Im } f(z) = (\text{Re } f(z))^2$, $Z \in D$. Show that f(z) is constant in D. **[10]**

❖ Show that an isolated singular point z_0 of a function f(z) is a pole of order m if and only if f(z) can be written in the form $f(z)=\frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0.
Moreover $\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 1$. **[15]**

❖ Evaluate the integral $\int_C \text{Re}\left(z^2\right)dz$ from 0 to 2 + 4i along the curve C where C is a parabola $y = x^2$. **[10]**

❖ Obtain the first three terms of Laurent series expansion of the function $f(z)=\frac{1}{(e^z-1)}$ about the point z = 0 valid int he region $0 < |z| < 2\pi$. **[10]**

2018

1. Prove that the function: $u(x,y)=(x-1)^3-3xy^2+3y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z. **(10)**

3. Show by applying the residue theorem that $\int_0^\infty \frac{dx}{\left(x^2+a^2\right)^2}=\frac{\pi}{4a^3}, a>0.$ **(15)**

4. Find the Laurent's series which represent the function $\frac{1}{(1+z^2)(z+2)}$ when
(i) $|z|<1$
(ii) $1<|z|<2$
(iii) $|z|>2$ **(15)**

2017

❖ Using contour integral method, prove that $\int_0^\infty \frac{x\sin mx}{a^2+x^2}dx=\frac{\pi}{2}e^{-ma}$ **(15)**

❖ For a function $f:\mathbb{C}\to\mathbb{C}$ and $n \geq 1$, let $f^{(n)}$ denote the n^{th} derivative of f and $f^{(0)} = f$. Let f be an entire function such that for some $n \geq 1$, $f^{(n)}\left(\frac{1}{k}\right)=0$ for all k = 1, 2, 3,....... Show that f is a polynomial. **(15)**

❖ Let f = u + iv be an analytic function on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$ at all points of D. **(15)**

❖ Determine all entire functions f(z) such that 0 is a removable singularity of $f\left(\frac{1}{z}\right)$. **(10)**

2016

❖ Is $v(x, y) = x^3 - 3xy^2 + 2y$ a harmonic function ? Prove your claim. If yes, find its conjugate harmonic function u(x, y) and hence obtain the analytic function whose real and imaginary parts are u and v respectively. **(10)**

❖ Let $\gamma:[0,1]\to\mathbb{C}$ be the curve
$\gamma(t)=e^{2\pi i t}, 0\leq t\leq 1$.

Find, giving justifications, the value of the contour integral

$$\int_{\gamma} \frac{dz}{4z^2 - 1} \qquad \textbf{(15)}$$

❖ Prove that every power series represents an analytic function inside its circle of convergence. **(20)**

2015

❖ Show that the function $v(x, y) = 1n\,(x^2 + y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x, y)$. Also find the corresponding analytic function $f(z) = u + iv$ in terms of z.

❖ Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \dfrac{2z-3}{z^2-3z+2}$ about the point $z = 0$.

❖ State Cauchy's residue theorem. Using it, evaluate the integral $\displaystyle\int_C \frac{e^z+1}{z(z+1)(z-i)^2}\,dz; C: |z| = 2$.

2014

❖ Prove that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, z \neq 0; f(0) = 0$$

satisfies Cauchy-Riemann equations at the origin, but the derivative of f at $z = 0$ does not exist.

❖ Expand in Laurent series the function

$$f(z) = \frac{1}{z^2(z-1)} \text{ about } z = 0 \text{ and } z = 1.$$

❖ Evaluate the integral $\displaystyle\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos,\right)^2}$ using residues.

2013

❖ Prove that if $b\,e^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - b\,e^z$ has n zeroes in the unit circle.

❖ Using Cauchy's residue theorem, evaluate the integral

$$I = \int_0^{\pi} \sin^4\theta\, d\theta$$

2012

❖ Show that the function defined by

$$f(z) = \begin{cases} \dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin. **(12)**

❖ Use Cauchy integral formula to evaluate $\displaystyle\int_C \frac{e^{3z}}{(z+1)^4}\,dz$, where C is the circle $|z| = 2$. **(15)**

❖ Expand the function $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent series valid for

(i) $1 < |z| < 3$

(ii) $|z| > 3$

(iii) $0 < |z+1| < 2$

(iv) $|z| < 1$ **(15)**

❖ Evaluate by Contour integration

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}, a^2 < 1. \qquad \textbf{(15)}$$

2011

❖ Evaluate by Contour integration, $\displaystyle\int_0^1 \frac{dx}{\left(x^2-x^3\right)^{1/3}}$. **(15)**

❖ Find the Laurent Series for the function $f(z) = \dfrac{1}{1-z^2}$ with centre z=1. **(15)**

❖ Show that the series for which the sum of first n terms $f_n(x) = \dfrac{nx}{1+n^2x^2}, 0 \le x \le 1$ cannot be differentiated term-by-term at x=0. What happens at $x \neq 0$? **(15)**

❖ If f(z)=u+iv is an analytic function of z=x+iy and $u - v = \dfrac{e^y - \cos x + \sin x}{\cos hy - \cos x}$, find f(z) subject to the condition, $f\left(\dfrac{\pi}{2}\right) = \dfrac{3-i}{2}$. **(12)**

2010

❖ Show that $u(x, y) = 2x - x^3 + 3xy^2$ is a harmonic function. Find a harmonic conjugate of u(x, y). Hence find the analytic function f for which u(x, y) is the real part. **(12)**

❖ (i) Evaluate the line integral $\int_C f(z)dz$. Where $f(z) = z^2$, c is the boundary of the triangle with vertices A (0, 0), B (1, 0), C (1, 2) in that order.
(ii) Find the image of the finite vertical strip R : x = 5 to x =9, $-\pi \le y \le \pi$ of z – plane under the exponential function. **(15)**

❖ Find the Laurent series of the function

$$f(z) = \exp\left[\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right] \text{ as } \sum_{n=-\infty}^{\infty} c_n z^n$$

for $0 < |z| < \infty$

Where $C_n = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - \lambda \sin\phi)d\phi, n = 0, \pm 1, \pm 2, \ldots,$

with λ a given complex number and taking the unit circle C given by $z = e^{i\phi}(-\pi \le \phi \le \pi)$ as contour in this region. **(15)**

2009

❖ Let $f(z) = \frac{a_0 + a_1 z + ... + a_{n-1}z^{n-1}}{b_0 + b_1 z + ... + b_n z^n}, b_n \ne 0.$,

Assume that the zeroes of the denominator are simple. Show that the sum of the residues of f(z) at its poles is equal to $\frac{a_{n-1}}{b_n}$. **(12)**

❖ If α, β, γ are real numbers such that $\alpha^2 > \beta^2 + \gamma^2$

Show that:

$$\int_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}}$$ **(30)**

2008

❖ Find the residue of $\frac{\cot z \cot hz}{z^3}$ at z = 0. **(12)**

❖ Evaluate

$$\int_C \left[\frac{e^{2z}}{z^2(z^2 + 2z + 2)} + \log(z-6) + \frac{1}{(z-4)^2}\right]dz$$

where c is the circle |Z|=3. State the theorem you use in evaluating above integral. **(15)**

❖ Let f(z) be entire function satisfying $f(z) \le k|z|^2$ for some +ve constant k and all z. show that $f(z) = az^2$ for some constant a. **(15)**

2007

❖ Prove that the function f defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, z \ne 0 \\ 0, z = 0 \end{cases}$$ is not differentiable at z = 0

(12)

❖ Evaluate (by using residue theorem)

$$\int_0^{2\pi} \frac{d\theta}{1 + 8\cos^2\theta}.$$ **(15)**

2006

❖ With the aid of residues, evaluate

$$\int_0^{\pi} \frac{\cos 2\theta d\theta}{1 - 2a\cos\theta + a^2}; -1 < a < 1.$$ **(15)**

❖ If f (z) = u + iv is an analytic function of the complex variable z and $u - v = e^x(\cos y - \sin y)$ determine f(z) in terms of z. **(12)**

❖ Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's series which is valid for (i) 1< |z| <3 (ii) |z| >3 (iii) |z| < 1. **(30)**

2004

❖ If all zeros of a polynomial p(z) lie in a half plane then show that zeros of the derivative p'(z) also lie in the same half plane. **(15)**

❖ Using contour integration , evaluate

$$\int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{1 - 2p\cos 2\theta + p^2}, 0 < p < 1.$$ **(15)**

2003

❖ Use the method of contour integration to prove that

$$\int_0^{\pi} \frac{a d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}; (a > 0).$$ **(15)**

2002

❖ Suppose that f and g are two analytic functions on the set $\mathbb{C}$ of all complex numbers with $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for n=1,2,3,….. then show that f(z)=g(z) for each z in $\mathbb{C}$. **(12)**

❖ Show that when $0 < |z-1| < 2$, the function $f(z) = \dfrac{z}{(z-1)(z-3)}$ has the Laurent series expansion in powers of z-1 as

$$-\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}. \qquad \textbf{(15)}$$

2001

❖ Prove that the Riemann Zeta function ζ defined by $\xi(z) = \sum_{n=1}^{\infty} n^{-z}$ converges for Re z > 1 and converges uniformly for $\operatorname{Re} z \geq 1 + \in$ where $\in > 0$ is arbitrary small. **(12)**

❖ Show that $\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx = \frac{\pi}{\sqrt{2}}$.

2000

❖ Suppose $f(\xi)$ is continuous on a circle C. show that $\int_C \frac{f(\xi)}{(\xi - z)}d\xi$ as z varies inside of 'C', is differentiable under the integral sign. Find the derivative hence or otherwise derive an integral representation for $f'(z)$ if $f(z)$ is analytic on and inside of C. **(30)**

1999

❖ Examine the nature of the function $f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}, z \neq 0$ $f(0) = 0$ in a region including the origin and hence show that Cauchy – Riemann equations are satisfied at the origin but f(z) is not analytic there.

❖ For the function $f(z) = \frac{-1}{z^2-3z+2}$, find Laurent series for the domain (i) $1 < |z| < 2$ (ii) $|z| > 2$ show further that $\oint_c f(z)\,dz = 0$ where 'c' is any closed contour enclosing the points z=1 and z=2 .

❖ Using residue theorem show that $\int_{-\infty}^{\infty}\frac{x\sin ax}{x^4+4}dx = \frac{\pi}{2}e^{-a}\sin a; (a>0)$ ***(1984, 1998)***

❖ The function f(z) has a double pole at z=0 with residue 2, a simple pole at z=1 with residue 2, is analytic at all other finite points of the plane and is bounded as $|z| \to \infty$. If f(2)=5 and f(-1) = 2, find f(z).

❖ What kind of singularities the following functions have?

(i) $\frac{1}{1-e^z}$ $at\ z = 2\pi i$

(ii) $\frac{1}{\sin z - \cos z}$ $at\ z = \frac{\pi}{4}$

(iii) $\frac{\cot \pi z}{(z-a)^2}$ $at\ z = a\ and\ z = \infty$.

In case (iii) above what happens when 'a' is an integer.(including a = 0)?

1998

❖ Show that the function $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, z \neq 0$

❖ $f(0) = 0$ is continuous and C-R conditions are satisfied at z=0, but $f'(z)$ does not exist at z=0.

❖ Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity $z = -2$. Specify the region of convergence and the nature of singularity at z = -2

❖ By using the integral representation of $f^n(0)$, prove that $\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_c \frac{x^n e^{xz}}{n!z^{n+1}}dz$, where 'c' is any closed contour surrounding the origin. Hence show that $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta}d\theta.$

❖ Using residue theorem $\int_0^{2\pi}\frac{d\theta}{3-2\cos\theta+\sin\theta}$.

1997

❖ If $f(z) = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \cdots + \frac{A_n}{(z-a)^n}$ find the residue at a for $\frac{f(z)}{z-b}$ where

$A_1, A_2, ...A_n$, a & b are constant. What is the residue at infinity.

❖ Find the Laurent series for the function $e^{1/z}$ in $0 < |z| \le \infty$.

Deduce that $\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{1}{n!}$

(n = 0, 1, 2, ...) **(2001)**

❖ Find the function f(z) analytic with in the unit circle which takes the values $\frac{a-\cos\theta+i\sin\theta}{a^2-2a\cos\theta+1}, 0\le\theta\le 2\pi$ on the circle.

❖ Integrating e^{-z^2} along a suitable rectangular contour. Show that $\int_0^{\infty} e^{-x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$.

1996

❖ Evaluate $\underset{z\to 0}{Lt}\frac{1-\cos z}{\sin(z^2)}$

❖ Show that z = 0 is not a branch point for the function $f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$. Is it a removable singularity ?

❖ Prove that every polynomial equation $a_0 + a_1 z + a_2 z^2 + ... + a_n z^n = 0, a_n \ne 0, n \ge 1$ has exactly 'n' roots.

❖ By using residue theorem, evaluate $\int_0^{\infty}\frac{\log_e(x^2+1)}{x^2+1}dx$

❖ About the singularity z =-2, find the Laurent expansion of $(z-3)\sin\frac{1}{z+2}$. Specify the region of convergence and nature of singularity at z = –2.

1995

❖ Let $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$. Prove that 'u' is a harmonic function. Find a harmonic function v such that u+iv is an analytic function of z.

❖ Find the Taylor series expansion of the function $f(z) = \frac{z}{z^4+9}$ around z = 0. Find also the radius of convergence of the obtained series.

❖ Let 'C' be the circle | Z |=2 described contour clockwise .Evaluate the integral $\int_c \frac{\cosh \pi z}{z(z^2+1)}dz$

❖ Let $a \ge 0$. Evaluate the integral $\int_0^{\infty}\frac{\cos ax}{x^2+1}dx$ with the aid of residues. **(2006)**

❖ Let f be analytic in the entire complex plane. Suppose that there exists a constant A > 0 such that $|f(z)| \le A|z|$ for all z. Prove that there exists a complex number 'a' such that f(z) = az for all z.

❖ Suppose a power series $\sum_{n=0}^{\infty} a_n z^n$ converges at a point $z_0 \ne 0$.

Let z_1 be such that $|z_1| < |z_0|$ and $z_1 \ne 0$

$|z_1| < |z_0|$ and $z_1 \ne 0$ show that the series converges uniformly in the disc $\{z : |z| \le |z_1|\}$.

1994

❖ How many zeros does the polynomial $p(z) = z^4 + 2z^3 + 3z + 4$. Posses in (i) the first quadrant (ii) the fourth quadrant.

❖ Test for uniform convergence in the region |Z| ≤ 1 the series $\sum_{n=1}^{\infty}\frac{\cos nz}{n^3}$.

❖ Find Laurent series for (i) $\frac{e^{2z}}{(z-1)^3}$ about z=1.

(ii) $\frac{1}{z^2(z-3)^2}$ about z = 3.

❖ Find the residues of $f(z) = e^z \operatorname{cosec}^2 z$ at all its poles in the finite plane.

❖ By means of contour integration, evaluate $\int_0^{\infty}\frac{(\log_e u)^2}{u^2+1}du$.

1993

❖ In the finite Z- plane show that the function

$f(z) = \sec\frac{1}{z}$.

has infinitely many isolated singularities in a finite interval which includes '0'.

❖ Prove that (by applying Cauchy integral formula or otherwise)

$$\int_0^{2\pi} \cos^{2n}\theta d\theta = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}2\pi,$$

$where\ n = 1, 2, 3, \ldots$

❖ If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points (1,1) and (2,3) find the value of $\int_C (12z^2 - 4iz)\,dz$

❖ Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converges absolutely for $|z| \leq 1$.

❖ Evaluate $\int_0^{\infty} \frac{dx}{x^6+1}$ by choosing an appropriate contour.

1992

❖ If $u = e^{-x}$ (x siny-ycosy), find 'v' such that f(z)= u + iv is analytic. Also find f(z) explicitly as a function of z. **(1997)**

❖ Let f(z) be analytic inside and on the circle C defined by |z| = R and let $z = re^{i\theta}$ be any point inside C. prove that

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(R^2 - r^2\right) f\left(\mathrm{Re}^{i\phi}\right)}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi .$$

❖ Prove that all roots of $z^7 - 5z^3 + 12 = 0$ lies between the circles $|z| = 1\ and\ |z| = 2$.

(1998,2006)

❖ Find the region of convergence of the series whose n-th term is $\frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$

❖ Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for

$(i)|z| > 3 \quad (ii) 1 < |z| < 3 \quad (iii)|z| < 1$ **(2005)**

❖ By integrating along a suitable contour evaluate $\int_0^{\infty} \frac{\cos mx}{x^2+1} dx$.

1991

❖ **A** function f(z) is defined for finite values of z by f(0) = 0 and $f(z) = e^{-z^{-4}}$ everywhere else. Show that the Cauchy Riemann equation are satisfied at the origin. Show also that f(z) is not analytic at the origin.

❖ If $|a| \neq R$ show that $\int_{|z|=R} \frac{|dz|}{|z-a||z+a|} < \frac{2\pi R}{\left|R^2 - |a|^2\right|}$

❖ If $J_n(t) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - t\sin\theta) d\theta.$ show that

$$e^{\frac{1}{2}\left(z - \frac{1}{z}\right)} = J_0(t) + zJ_1(t) + z^2 J_2(t) + ---$$

$$-\frac{1}{z}J_1(t) + \frac{1}{z^2}J_2(t) - \frac{1}{z^3}J_3(t) + ---$$

❖ Examine the nature of the singularity of e^z at infinity

❖ Evaluate the residues of the function $\frac{Z^3}{(Z-2)(Z-3)(Z-5)}$ at all singularities and show that their sum is zero.

❖ By integrating along a suitable contour show that $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} = \frac{\pi}{\sin a\pi}$ where 0 < a < 1.

1990

❖ Let f be regular for |Z| < R , prove that , if 0<r<R,

$$f'(0) = \frac{1}{\pi r}\int_0^{2\pi} u(\theta) e^{-i\theta} d\theta ;$$

where $u(\theta) = \mathrm{Re}\ f(re^{i\theta})$

❖ Prove that the distance from the origin to the nearest zero of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is at least $\frac{r|a_0|}{M + |a_0|}$. where r is any number not exceeding the radius of the convergence of the series and $M = M(r) = \sup_{|z|=r} |f(z)|$.

❖ Prove that $\int_{-\infty}^{\infty} \frac{x^4}{1+x^8} dx = \frac{\pi}{\sqrt{2}} \sin\frac{\pi}{8}$ using residue calculus.

❖ Prove that if f = u + iv is regular through out the complex plane and au + bv-c ≥ 0 for suitable constants a,b,c then f is constant.

- Derive a series expansion of $\log(1+e^z)$ in powers of z.
- Determine the nature of singular points $\sin\left(\frac{1}{\cos 1/z}\right)$ and investigate its behaviour at $z=\infty$.

1989

- Find the singularities of $\sin\left(\frac{1}{1-z}\right)$ in the complex plane.

1988

- By evaluating $\int\frac{dz}{z+2}$ over a suitable contour C, Prove that $\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}d\theta$ **(1997)**
- If f is analytic in $|Z|\le R$ and x, y lie inside the disc, evaluate the integral $\int_{|z|=R}\frac{f(z)dz}{(z-x)(z-y)}$ and deduce that a function analytic and bounded for all finite z is a constant.
- If $f(z)=\sum_{n=0}^{\infty}a_nz^n$ has radius of convergence R and prove that $\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^2 d\theta=\sum_{n=0}^{\infty}|a_n|^2 r^{2n}$
- Evaluate $\int_C\frac{Ze^z}{(z-a)^3}$, if a lies inside the closed contour C.
- Prove that $\int_0^{\infty}e^{-x^2}\cos(2bx)\,dx=\frac{\sqrt{\pi}}{2}e^{-b^2};(b>0)$ by the integrating e^{-Z^2} along the boundary of the rectangle |x| d" R,0 d" y d"b. **(1997)**
- Prove that the coefficients C_n of the expansion $\frac{1}{1-z-z^2}=\sum_{n=0}^{\infty}C_nz^n$ satisfy $C_n=C_{n-1}+C_{n-2}$, $n\ge 2$ Determine C_n.

1987

- By considering the Laurent series for $f(z)=\frac{1}{(1-z)(z-2)}$ prove that if 'C' be a closed contour oriented in the contour clockwise direction, then $\int_C f(z)\,dx=2\pi i$
- State and prove Cauchy's residue theorem.
- By the method of contour integration , show that $\int_0^{\infty}\frac{\cos x}{x^2+a^2}dx=\frac{\pi e^{-x}}{2a}, a>0.$

1986

- Let f(z) be single valued and analytic with in and on a closed curve C. If z_0 is any point interior to C, then show that $f(z_0)=\frac{1}{2\pi i}\int_c\frac{f(z)}{z-z_0}dz$,where the integral is taken in the +ve sense around C.
- By contour integration method show that

  (i) $\int_0^{\infty}\frac{dx}{x^4+a^4}=\frac{\pi\sqrt{2}}{4a^3}$, where a > 0.

  (ii) $\int_0^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$.

1985

- Prove that every power series represents an analytic function within its circle of convergence.
- Prove that the derivative of a function analytic in a domain is itself an analytic function.
- Evaluate , by the method of contour integration $\int_0^{\infty}\frac{x\sin ax}{x^2-b^2}dx$.

1984

- Evaluate by contour integration method :

  (i) $\int_0^{\infty}\frac{x\sin mx}{x^4+a^4}dx$

  (ii) $\int_0^{\infty}\frac{x^{a-1}\log x}{1+x^2}dx$ **(1998, 1999)**
- Distinguish clearly between a pole and an essential singularity. If z = a is an essential singularity of a function f(z) , then for an arbitrary positive integers $\eta,\in and\ \rho$, prove that $\exists$ a point z , such that $0<|z-a|<\rho\ for\ which\,|f(z)-\eta|<\in$.

1983

- Obtain the Taylor and Laurent series expansions which represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the regions (i) |z| < 2 (ii) 2 < |z| < 3 (iii) |z| > 3.

❖ Use the method of contour integration to evaluate

$$\int_0^\infty \frac{x^{a-1}}{1+x^2} ü \; ü < \quad <$$

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IFoS
PREVIOUS YEARS QUESTIONS (2020-2000)
SEGMENT-WISE

COMPLEX ANALYSIS

(ACCORDING TO THE NEW SYLLABUS PATTERN) PAPER - II

2020

❖ Evaluate the integral $\int_C \mathrm{Re}\left(z^2\right)dz$ from 0 to 2 +4i along the curve C : y = x^2. **[08]**

❖ Using Cauchy theorem and Cauchy integral formula, evaluate the integral

$$\oint_C \frac{e^z}{z^2\left(z+1\right)^3}dz \text{ where C is } |z| = 2.$$ **[15]**

❖ Show that the bilinear transformation

$$w = e^{i\theta_0}\left(\frac{z-z_0}{z-\overline{z}_0}\right)$$

Z_0 being in the upper half of the z-plane, maps the upper half of the z-plane into the interior of the unit circle in the w-plane. If under this transformation, the point z = i is mapped into w = 0 while the point at infinity is mapped into w = –1, then find this transformation. **[10]**

2019

❖ Using Cauchy's Integral formula, evaluate the integral $x^2 + y^2 + z^2 = a^2$ where c : |z – i| = 2. **(08)**

❖ If f(z) is analytic in a domain D and |f(z)| is a non-zero constant in D, then show that f(z) is constant in D. **(15)**

❖ Classify the singular point z = 0 of the function $a^2 + b^2 = 6c^2$. and obtain the principal part of the Laurent series expansion of f(z). **(15)**

2018

❖ If $u = (x - 1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x+iy$ **(10)**

❖ Evaluate the integral $\int_0^{2\pi}\cos^{2n}\theta\, d\theta$, where n is a positive integer. **(10)**

2017

❖ If f(z) = u(x, y) + iv(x, y) is an analytic function of z = x+ iy and u + 2v = $x^3 - 2y^3 + 3xy\,(2x- y)$ then find f(z) in terms of z. **(8)**

❖ Prove by the method of contour integration that $\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}d\theta = 0.$ **(10)**

❖ Find the sum of residues of $f(z) = \dfrac{\sin z}{\cos z}$ at its poles inside the circle |z| = 2. **[8]**

2016

❖ Find the analytic function of which the real part is $e^{-x}\left\{\left(x^2-y^2\right)\cos y + 2xy\sin y\right\}$. **(8)**

❖ Find the Laurent series for the function $f(z) = \dfrac{1}{1-z^2}$ with centre z = 1. **(10)**

❖ Evaluate by Contour integration $\int_0^{\pi}\dfrac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$. **(10)**

2015

❖ Let u(x, y) = cos x sinh y. Find the harmonic conjugate v(x, y) of u and express u(x, y) + i v (x, y) as a function of z = x + iy. **(8)**

❖ Evaluate the integral $\int_r \frac{z^2}{(z^2+1)(z \square 1)^2}dz$, where r is the circle |z| = 2 **(12)**

❖ Show that $\int_{\square\infty}^{\infty}\frac{x^2}{1+x^4}dx = \frac{\pi}{\sqrt{2}}$ by using contour integration and the residue theorem. **(15)**

2014

❖ Using Cauchy integral formula, evaluate

$$\int_C \frac{z+2}{(z+1)^2(z \square 2)}dz$$

where C is the circle $|z–i| = 2$

- Find the constants a, b, c such that the function $f(z)=2x^2-2xy-y^2+i(ax^2-bxy+cy^2)$ is analytic for all $z(=x+iy)$ and express $f(z)$ in terms of z. **(8)**
- Evaluate :

$$\int_C \frac{z}{z^4-6z^2+1}dz$$

when C is the circle $|z-i|=2$ **(8)**
- Find the bilinear transformation which map the points $-1, \infty, i$ into the points–
  (i) i, 1, 1 + i
  (ii) ∞, i, 1
  (iii) 0, ∞, 1 **(15)**
- Find the Laurent series expansion at z=0 for the function

$$f(z)=\frac{1}{z^2(z^2+2z-3)}$$

in the regions *(i)* $1<|z|<3$ and *(ii)* $|z|>3$. **(15)**

2013

- Construct an analytic function $f(z)=u(x,y)+iv(x,y)$, where $v(x,y)=6xy-5x+3$. Express the result as a function of z.
- Evaluate $\oint_c \frac{e^{2z}}{(z+1)^4}dz$ where c is the circle $|z|=3$.
- Find Laurent series about the indicated singularity. Name the singularity and give the region of convergence.

$$\frac{z-\sin z}{z^3}; z=0.$$

2012

- Evaluate the integral

$$\int_{2-i}^{4+i}(x+y^2-ixy)dz$$

along the line segment AB joining the points A(2,–1) and B(4, 1). **(10)**
- Show that the function $u(x,y)=e^{-x}(x\cos y+y\sin y)$ is harmonic. Find its conjugate harmonic function $v(x,y)$ and the corresponding analytic function f(z). **(13)**
- Using the Residue Theorem, evaluate the integral

$$\int_C \frac{e^z-1}{z(z-1)(z+i)^2}dz,$$

where C is the circle $|z|=2$ **(13)**

2011

- Expand the function

$$f(z)=\frac{2z^2+11z}{(z+1)(z+4)}$$

in a Laurent's series valid for $2<z<3$. **(10)**
- Examine the convergence of $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ and evaluate, if possible. **(10)**
- State Cauchy's residue theorem. Using it, evaluate the integral

$$\int_C \frac{e^{z/2}}{(z+2)(z^2-4)}dz$$

Counterclockwise around the circle C:$|z+1|=4$. **(13)**

2010

- Determine the analytic function $f(z)=u+iv$ if $v=e^x(x\sin y+y\cos y)$ **(10)**
- Using the method of contour integration, evaluate

$$\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+1)^2(x^2+2x+2)}$$ **(14)**
- Obtain Laurent's series expansion of the function $f(z)=\frac{1}{(z+1)(z+3)}$ in the region $0<|z+1|$ Z< 2 (13)

2009

- Evaluate

$$\int_C \frac{2z+1}{z^2+z}dz$$

By Cauchy's integral formula, where C is $|z|=\frac{1}{2}$ **(10)**
- Determine the analytic function $w=u+iv$, is

$$u=\frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$$ **(13)**
- Evaluate by contour integration

$$\int_0^{2\pi}\frac{d\theta}{1-2a\sin\theta+a^2},\ 0<a<1$$ **(13)**

2008

- Evaluate $\int_C \bar{z}\,dz$ from $z=0$ to $z=4+zi$. Along the curve given by $z=t^2+it$. **(10)**

❖ Expand in a Laurent's series the function $f(z)=\frac{1}{(z-1)z^2}$ about z = 0. **(13)**

❖ Find the residue of $f(z) = \tan z$ at $\pi/2$. **(13)**

2007

❖ If $f(z) = u + iv$ is analytic and $u=e^{-x}(x\sin y - y\cos y)$ then find v and f(z). **(10)**

❖ Applying Cauchy's criterion for convergence, show that the sequence (S_n) defined by $S_n = 1+1/2+1/3+....+$ is not convergent. **(13)**

❖ Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in a Laurent series valid for (i) $1<|z|>3$. (ii) $|z|>3$. **(13)**

❖ Using residue theorem, evaluate

$$\int_0^{2\pi}\frac{d\theta}{(3-2\cos\theta+\sin\theta)}$$

2005

❖ If f is analytic, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$$ **(10/2006)**

❖ Show that the transformation $w=\frac{5-4z}{4z-2}$ maps unit circle |z| = 1 onto a circle of radius unity and centre at $-1/2$ **(13/2006)**

❖ Use contour integration technique to find the value of $\int_0^{2p}\frac{d\theta}{2+\cos\theta}$ **(14/2006)**

2004

❖ Investigate the continuity at (0, 0) of the function

$$f(x,y)=\begin{cases}\frac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$ (10)

❖ Find the analytic function $f(z)=u(x,y)+iv(x,y)$ for which $u-ve^x(\cos y-\sin y)$.

❖ Find the bilinear transformation that maps $z=1,0,\infty$ to $w=0,-\infty,1$ respectively. (13)

❖ Find the singular points with their nature and the residues there at of $f(z)=\frac{\cot\pi z}{(z-1/3)^2}$ (13)

❖ Prove that a function analytic for all finite values of z and bounded, is a constant. (13)

2003

❖ If $w=f(z)=u(x,y)+iv(x,y), z=x+iy,$ is analytic in a domain, show that $\frac{\partial w}{\partial \bar{z}}=0$. Hence or otherwise, show that $\sin(x+i3y)$ cannot be analytic

❖ Discuss the transformation $w=z+\frac{1}{z}$ and hence, show that

(1) a circle in z-plane is mapped on an ellipse in the w-plane

(2) a line in the z-plane is mapped into a hyperbola in the w-plane. **(13)**

❖ Find the Laurent series expansion of the function $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ Valid in the region $2<|z|<3$. **(13)**

2002

❖ If $f(z)$ has a simple pole with residue K at the origin and is analytic on $0<|z|\leq|$ Show that

$$\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-a)(z-b)}dz=\frac{f(a)-f(b)}{a-b}+\frac{K}{ab}$$

Where $0<a, b, <1$ and C is the circle $|z| = 1$.

❖ If $f(a)=\oint_C\frac{3z^2+7z+1}{z-a}dz$ Where C is the circle $|z| = 2$; Find (i) $f(1-i)$; (ii) $f''(1-i)$; (iii) $f(1+i)$ **(12)**

❖ Under the bilinear transformation $w = \frac{3-z}{z-2}$

Find the images of

(1) $\left|z - \frac{5}{2}\right| = 1/2$ *and*

(2) $\left|z - \frac{5}{2}\right| < 1/2$ *in the W – plane.*

2001

❖ Compute the Taylor series around z = 0 and give the radius of convergence for $\frac{z}{z-1}$

❖ Show that the function $f(z) = \sqrt{xy}$ is not regular at the origin although the Cauchy-Riemann equations are satisfied **(13)**

❖ By using the Residue Theorem evaluate the integral $\int_0^{2\pi} \frac{d\theta}{1 - 2a\sin\theta + a^2}$ *Where* $0 < a < 1.$ **(14)**

2000

❖ Expand the function $f(z) = \log(z+2)$ in a power series and determine its radius of convergence.

❖ Prove that the function $f(z) = u + iv$

Where $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$

$f(0) = 0$

Satisfies Cauchy-Riemann equations at the origin, but $f'(0)$ does not exist.

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