Date : 12/12/2020

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

145/250

MAINS TEST SERIES-2020

(OCT. TO JAN.-2020-21)

IAS/IFoS

MATHEMATICS

Under the guidance *of* K. Venkanna

Common Test
Test-16 for Batch-I
&
Test-8 for Batch-II

FULL SYLLABUS (PAPER-II)

DATE : 06-DEC.-2020

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 35 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side.. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Vinayak Namsade

Roll No.:

Test Centre: ONLINE

Medium: ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate: Vinayak

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			06	
	(b)			06	
	(c)			05	
	(d)			02	
	(e)				19
2	(a)				
	(b)				
	(c)				
	(d)				
3	(a)				
	(b)				
	(c)				
	(d)				
4	(a)			09	
	(b)			10	
	(c)			05	
	(d)			03	27
5	(a)			07	
	(b)			00	
	(c)			09	
	(d)			07	
	(e)			03	26
6	(a)			08	
	(b)			04	
	(c)			12	
	(d)			15	39
7	(a)			02	
	(b)			09	
	(c)			11	
	(d)			12	34
8	(a)				
	(b)				
	(c)				
	(d)				
Total Marks					145

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SECTION – A

1. (a) Show that A_4 does not contain a subgroup of order 6. **[10]**

$A_4 = \{$ group of even transposition on $(1\ 2\ 3\ 4)\}$.

$\therefore O(A_4) = \frac{4!}{2} = 12$

Suppose $O(H) = 6$ $\quad H < A_4$

From Lagrange's formula $6 \mid 12$.

Let $C = \{(1\ 2\ 3)\ (1\ 3\ 2)\ (2\ 3\ 4)\ (2\ 4\ 3)\ (1\ 3\ 4)\ (1\ 4\ 3)\ (1\ 2\ 4)\ (1\ 4\ 2)\}$

$C \subseteq A_4$.

$\frac{O(A_4)}{O(H)} = 2 \Rightarrow A_4 = H \cup Ha = H \cup aH$

$a \notin H$, $a \in A_4$

$\therefore (1\ 2\ 3) \in A_4$ $\quad Ha = aH$.

$\therefore (1\ 2\ 3)H = H(1\ 2\ 3)$

By this sense $\forall\ a \in C$, $a \in H$.

$\therefore O(H) \geq 8$ which is contradiction

$\therefore$ H cannot be a subgroup of order 6.

06

1. (b) Prove that $x^3 - 9$ is irreducible over the integers mod 31. [10]

$f(x) = x^3 - 9$ let $Z_{31} = \{0, 1, 2 \ldots 30\}$

~~There~~ If there exists no solution for $f(x)$ in Z_{31} then $x^3 - 9$ is irreducible

$f(x) = 0$ if $x^3 - 9$

i.e $x \times_{31} x \times_{31} x = 9$

$\therefore$ ~~$x^3 \in 9, 40$~~ $x^3 = 31z + 9$ $z = 0, 1, 2 \ldots$

$x \in \{0, \ldots 30\}$

$x^3 = 31z + 9$??

$31 \mid (x^3 - 9)$

Please show it explicitly

$\nexists$ no $x \in \{0, 1 \ldots 30\}$ st $31 \mid x^3 - 9$

$\therefore f(x)$ is irreducible over Z_{31}.

06

1. (c) Define an open set. Prove that the union of an arbitrary family of open sets is open. Show also that the intersection of a finite family of open sets is open. Does it hold for an arbitrary family of open sets? Explain the reason for your answer by example. **[10]**

An open set is a set which consists of all elements in the interval except the boundary points. Ex. (a,b) ~~is open set~~. In this case every point of open set has a neighbourhood which is a subset of the given set.

Please consider Generic notation for open set.

$$\bigcup_{\text{infinite}} (a,b) = (a,b) \cup (c,d) \cup (e,f) \cdots \cup (g,h) \cup \cdots$$

$$= (\min(a,c,\cdots g), \max(b,d,\cdots h))$$

$$= (\alpha, \beta) \Rightarrow \text{which is open}$$

$$= (-\infty, \infty) \Rightarrow \text{if infinite} \Rightarrow \text{open}$$

because open interval $\Rightarrow$ open set

but open $\not\Rightarrow$ open interval

$$\bigcap_{\text{finite}} (a,b) = (a,b) \cap (c,d) \cap (e,f) \cdots \cap (g,h)$$

$$= \{\max(a,b,\cdots g), \min(b,d,f \cdots h)\}$$

$$= (\eta, \eta) \text{ open}$$

consider $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = (-1,1) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) \cdots \left(-\frac{1}{n}, \frac{1}{n}\right)$

$$= \{0\}$$

$\Rightarrow$ closed. Justify or simply state not open.

Intersection: Arbitrary family of open sets ~~is~~ is not open

05

1. (d) Find the regular function $w = u + iv$ where $u = e^x \{(x^2 - y^2) \cos y + 2xy \sin y\}$. [10]

$w = u + iv$.

$u = e^x[(x^2-y^2)\cos y + 2xy \sin y]$

Cauchy Riemmans equation for regular function

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\qquad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

-02

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x[2x\cos y + 2y \sin y] + e^x[(x^2-y^2)\cos y + 2xy \sin y)$

$v = e^x[2x\sin y + 2(-y\cos y + \sin y)] + e^x[(x^2-y^2)(\sin y) + [\cancel{+2y\sin y}] - 2y\cos y + 2\sin y]$

$+ e^x[2x][-y\cos y + \sin y] + \phi(x)$

$v = e^x[4x\sin y - 4y\cos y + 4\sin y + (x^2-y^2)\sin y - 2xy\cos y) + \phi(x)$

$\cancel{\frac{\partial v}{\partial y} = e^x[4x\cos y + 4y\sin y - 4\cos y + 4\cos y + (x^2-y^2)\cos y + 2xy\sin y - 2x\cos y)}$.

$\frac{\partial v}{\partial x} = e^x[4x\sin y - 4y\cos y + 4\sin y + (x^2-y^2)\sin y - 2xy\cos y) + \phi'(x)$

$+ e^x[4\sin y + 2x\sin y - 2y\cos y]$

$\frac{\partial u}{\partial y} = e^x[+x^2\sin y - y^2\sin y + 2y\cos y - 2xy\cos y - 2x\sin y]$

1. (e) Two products are manufactured sequentially on two machines. The time available on each machine in 8 hours per day and may be increased by up to 4 hours of overtime, If necessary at an additional cost of Rs. 100 per hour. The table below gives the production rate on the two machines as well as the price per unit of the two products. Determine the optimum production schedule and the recommended use of overtime if any.

	Production rate (units/hr)		
	Product 1	Product 2	Hours
Machine 1	5	5	8
Machine 2	8	4	8
Price per unit (Rs)	110	118	

[10]

To Min cost

Let x_1 x_2 be units produced of product 1 and 2 on machine ① and ②

~~Let y_1 y_2 be units produced of product~~

~~Let z =~~ Let t_1 and t_2 be overtime on machine ① and ② respectively.

Zero only

4. (a) If R is ring, let Z(R) = {x ∈ R | xy = yx all y ∈ R}. Prove that Z(R) is a subring of R. Is Z(R) an ideal ? If not, justify your answer. **[13]**

R is Ring.

$Z(R) = \{x \in R \mid xy = yx \ \forall\, y \in R\}$

Let $a, b \in Z(R)$ $\therefore ay = ya$ $\forall\, y \in R$

$by = yb$

Subgroup

$(a-b)y = ay - by = ya - yb = y(a-b)$

$\therefore (a-b) \in Z(R) \Rightarrow Z(R)$ is a subgroup of R

Closure $(ab)y = aby = ayb = y(ab)$

$\therefore ab \in Z(R) \Rightarrow$ closure satisfied

$\therefore Z(R)$ is a subring of R $\therefore$ By property $x \in Z(R)$, $x \in R$ satisfies all other conditions

Ideal $g \in R$, $a \in Z(R)$, $y \in R$

Let assume $g \notin Z(R)$

$(ag)y \neq ayg$ $\neq y(ag)$ $\Rightarrow$ Not right ideal

$\in Z(R)$

$(ga)y = gya \neq yga \notin Z(R) \Rightarrow$ Not left ideal.

09

4. (b) Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & , \text{if } x < 0 \\ c & , \text{if } x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{1/2}} & , \text{if } x > 0 \end{cases}$$

Determine the values of a, b, c for which the function is continuous at x = 0.

[12]

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{\sin(a+1)x + \sin x}{x}$$

$$= \lim_{x \to 0} \frac{\sin(a+1)x}{(a+1)x}(a+1) + \frac{\sin x}{x}$$

$$= a+1+1 = a+2$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$$

$$= \lim_{x \to 0} \frac{(1+bx)^{1/2} - 1}{bx^1} \quad \left(\frac{0}{0}\right)$$

Apply L'Hospital

$$= \lim_{x \to 0} \frac{\frac{\cancel{b}}{2\sqrt{1+bx}}}{\cancel{b}} = \frac{1}{2}$$

10

For continuity

$$\lim_{x \to 0^+} f(x) = f(0) = \lim_{x \to 0^-} f(x)$$

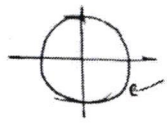
$$\therefore a+2 = c = \frac{1}{2}$$

$$\therefore a = -\frac{3}{2} \quad c = \frac{1}{2} \quad b \in \mathbb{R} - \{0\}$$

4. (c) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals.

(i) $\int_{|z|=1} \frac{z+3}{z^4+az^3}dz; (|a|>1)$ (ii) $\int_{|z|=4} \frac{z^4}{(z-i)^3}dz$ **[10]**

(i) $\int_{|z|=1} \frac{z+3}{z^3(z+a)}$



$|z|=1$

It has 2 poles $z=0$ of order 3
$z=-a$ of order 1 $\Rightarrow$ lies outside

Residue of $z=0$

$a_{-1} = \frac{1}{3!} \times \lim_{z\to 0} \frac{d^2}{dz^2} \frac{(z+3)}{(z+a)} = \left(1+\frac{3-a}{z+a}\right)$

$= \frac{1}{3!} \lim_{z\to 0} \frac{d}{dz}\left[\frac{-(3-a)}{(z+a)^2}\right]$

$a_{-1} = \frac{1}{3!} \lim_{z\to 0} \frac{2(3-a)}{(z+a)^3} = \frac{\cancel{2}(3-a)}{\cancel{3!}\,a^3}$ $\quad \frac{2\pi i(3-a)}{a^3}$

$a_{-1} = \frac{3-a}{3a^3}$

$\therefore$ By Cauchy's theorem, $\int_{|z|=1} \frac{z+3}{z^3(z+a)}dz = \frac{2\pi i(3-a)}{3a^3}$

(ii) $\int \frac{z^4}{(z-i)^3}$

$|z|=4$

pole i of order 3.

$a_{-1} = \frac{1}{3!} \lim_{z\to i} \frac{d^2}{dz^2}(z^4) = \frac{1}{3!} \lim_{z\to i} 12z^2$

$= \frac{-12}{6} = -2$

$\therefore \int_{|z|=4} \frac{z+3}{(z-i)^3} = 2\pi i(-2) = -4\pi i$

-05

$12\pi i$

4. (d) Solve the following transportation problem

		Destinations						
		D_1	D_2	D_3	D_4	D_5	D_6	*Availability*
Factories	F_1	2	1	3	3	2	5	50
	F_2	3	2	2	4	3	4	40
	F_3	3	5	4	2	4	1	60
	F_4	4	2	2	1	2	2	30
	Demand	30	50	20	40	30	10	180

by finding the initial solution by matrix minima method.

[15]

2 X	1 [50]	3 X	3 X	2 X	5 X	~~50~~ 0
3 [20]	2 X	2 [20]	4 X	3	4 X	~~40~~ ~~20~~ 0
3 [10]	5 X	4 X	2 [10]	4 [30]	1 [10]	~~60~~ ~~50~~ ~~40~~ ~~30~~ 0
4 X	2 X	2 X	1 [30]	2 X	2 X	~~30~~ 0
~~30~~ ~~10~~ 0	~~50~~ 0	~~20~~ 0	~~40~~ ~~10~~	~~30~~ 0	~~10~~ 0	

2 [0]	1 [50]	3	3	2	5
3 [20]	2	2 [20]	4	3	4
3 [10]	5	4	2 [10]	4 [30]	1 [10]
4	2	2	1 [30]	2	2

03

Given sol^n is degenerate $\because$ 8 entries

$m+n-1 = 6+4-1 = \underline{\underline{9}}$.

$\therefore$ Adding 0 in 1st cell.

Please proceed ahead

77.

SECTION – B

5. (a) Find the family orthogonal to $\phi[z(x+y)^2, x^2-y^2] = 0$ **[10]**

$$z(x+y)^2 = c(x^2-y^2)$$

$$\frac{z(x+y)^2}{x^2-y^2} = c$$

Diff. w.r.t x, y, z

$$f_x = z\left[\frac{2(x^2-y^2)(x+y)-(x+y)^2(2x)}{(x^2-y^2)^2}\right]$$

$$= \frac{2z(x+y)(x^2-y^2-2x^2-2xy)}{(x^2-y^2)^2}$$

$$= \frac{-2z(x+y)\cancel{(x+y)^2}}{\cancel{(x+y)^2}(x-y)^2} = \frac{-2z(x+y)}{(x-y)^2}$$

$$f_y = \frac{2z(x+y)}{(x-y)^2} \quad \text{By symmetry}$$

$$f_z = \frac{(x+y)^2}{x^2-y^2} = \frac{(x+y)}{(x-y)}$$

Orthogonal trajectories to (1) are

$$\frac{dx}{f_x} = \frac{dy}{f_y} = \frac{dz}{f_z} \Rightarrow \frac{dx}{\frac{-2z\cancel{(x+y)}}{(x-y)^{\cancel{2}}}} = \frac{dy}{\frac{2z\cancel{(x+y)}}{(x-y)^{\cancel{2}}}} = \frac{dz}{\frac{\cancel{(x+y)}}{\cancel{(x-y)}}}$$

$$dx = -dy \qquad \frac{dx-dy}{\frac{-4z}{(x-y)}} = \frac{dz}{1}$$

$$\boxed{x+y = c}$$

$$\therefore (x-y)\,d(x-y) = -4z\,dz$$

$$\frac{(x-y)^2}{2} = -2z^2 + d$$

$$\boxed{(x-y)^2 + 4z^2 = d}$$

$\therefore f(x+y, (x-y)^2+4z^2) = 0$ is required trajectory

05

07 @

Please refer 1001

5. (b) Find complete integral of $xp - yq = xqf(z - px - qy)$. [10]

$xp - yq = xq\, f(z - px - qy)$

$xp - q(y + x f(z - px - qy)) = 0$

Lagranges AE

$\frac{dx}{x} = \frac{dy}{-(y + x f(z - px - qy))} = \frac{dz}{0}$

$\frac{dy}{dx} = -\frac{y}{x} - f(z - px - qy)$ $\boxed{z = c}$

$f(z - px - qy) = \frac{xp - yq}{xq}$

Zero only

(c) The following are the number of deaths in four successive ten year age groups. By using Newton's forward formula find the number of deaths at 45-50 and 50-55.

Age Group	25-35	35-45	45-55	55-65
Deaths	13229	18139	24225	31496

[10]

Annulative table

Age	Deaths	Δy	$\Delta^2 y$	$\Delta^3 y$
35	13229			
		18139		
45	31368		6086	
		24225		1185
55	55593		7271	
		31496		
65	87089			

Newtons Forward formula

$$y(50) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$u = \frac{(50-35)}{10} = 1.5$$

$$y(50) = 13229 + 1.5(18139) + \frac{(1.5)(0.5)}{2!}(6086) + \frac{(1.5)(0.5)(-0.5)}{6}(1185)$$

$$y(50) = 42645.6875 \approx 42646$$

$\therefore$ No. of deaths betn 45-50 $= y(50) - y(45)$

$= 11277.6875$

≈ 11278

No. of deaths betn 50-55 $= y(55) - y(50)$

$= 12947.31$

≈ 12947

09

5. (d) Draw a switching circuit that realizes the following switching function. If possible, draw a simpler switching circuit. **[10]**

x	y	z	f(x,y,z)
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

Please draw switching circuit

$f = xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$

$= x\bar{z}(y+\bar{y}) + x\bar{y}z + \bar{x}\bar{z}(y+\bar{y})$ — Distribution

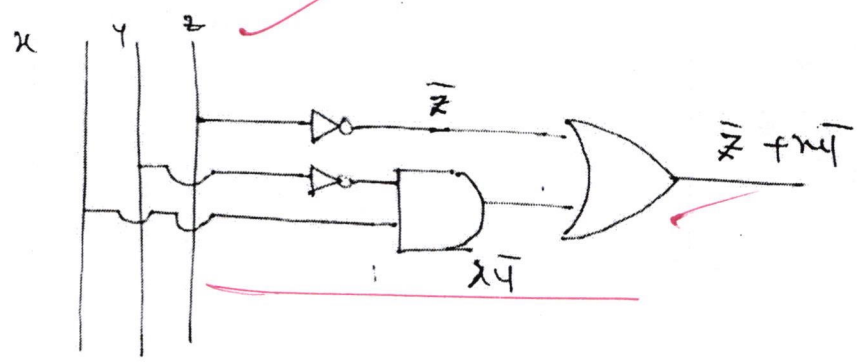
$= x\bar{z}(1) + x\bar{y}z + \bar{x}\bar{z}$ $\quad \because y+\bar{y}=1$

$= \bar{z}(x+\bar{x}) + x\bar{y}z$

$= \bar{z} + x\bar{y}z$

$= (\bar{z}+x\bar{y})\cdot(\bar{z}+z)$

$f = (\bar{z}+x\bar{y})$



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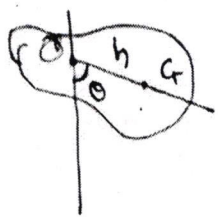
5. (e) Write the Hamiltonian function and equation of motion a compound pendulum. **[10]**



Let O be point of suspension G be CG and k be radius of gyration OG = h.

Let θ be θ at any time t.

$T = Mk^2\dot{\theta}^2$ $\qquad V = -Mh\cos\theta$

$L = T - V = Mk^2\dot{\theta}^2 + Mh\cos\theta$ is Lagrangian

$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = 2Mk^2\dot{\theta}$ which is Independent of t (time)

$\therefore H = T + V = \frac{p_\theta^2}{4Mk^2} - Mh\cos\theta$

$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -Mh\sin\theta$

$\dot{\theta} = \frac{+\partial H}{\partial p_\theta} = \frac{p_\theta}{2Mk^2}$

} Hamiltonian function

$p_\theta = -Mh\sin\theta \qquad p_\theta = 2Mk^2\dot{\theta}$

$-Mh\sin\theta = 2Mk^2\ddot{\theta}$

$\ddot{\theta} = \frac{-h}{2k^2}\sin\theta$

$$\boxed{\ddot{\theta} = \frac{-h}{2k^2}\theta}$$ if θ is small

03

6. (a) (i) Form a partial differential equation by eliminating the function ϕ from $lx + my + nz = \phi(x^2 + y^2 + z^2)$.

(ii) Solve $(x^3 + 3xy^2)\,p + (y^3 + 3x^2y)q = 2z(x^2 + y^2)$.

[5+7=12]

i) $lx + my + nz = \phi(x^2+y^2+z^2)$

Differentiating wrt x and y

$\therefore l + n\frac{\partial z}{\partial x} = \phi'(x^2+y^2+z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right)$ — ①

$m + n\frac{\partial z}{\partial y} = \phi'(x^2+y^2+z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right)$ — ②

Eliminating ϕ' from ① & ②

$\therefore \frac{(l+np)}{(m+nq)} = \frac{(x+zp)}{(y+zq)}$ — is required PDE

ii) It is of form of Lagranges Auxillary eqn.

$$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2z(x^2+y^2)}$$

Multiplying factors $(x, -y, 1)$

$$\frac{x\,dx - y\,dy}{x^4-y^4} = \frac{dz}{2z(x^2+y^2)} \Rightarrow \frac{2(x\,dx - y\,dy)}{x^2-y^2} = \frac{dz}{z}$$

$\ln(x^2-y^2) = \ln z + \ln c \Rightarrow (x^2-y^2) = zc$ — ①

$$\frac{dy}{dx} = \frac{y^3+3x^2y}{x^3+3xy^2} \Rightarrow y = vx$$

$$v + x\frac{dv}{dx} = \frac{v^3+3v}{1+3v^2} \Rightarrow x\frac{dv}{dx} = \frac{v^3+3v-v-3v^3}{1+3v^2}$$

08

$$x\frac{dv}{dx} = \frac{dx}{x} = \frac{1+3v^2}{2v-2v^3}\,dv \Rightarrow \frac{dx}{x} = \frac{3v^2-1}{2(v-v^3)}\,dv + \frac{1}{v-v^3}\,dv$$

$\ln x = -\frac{1}{2}\ln(v-v^3) + \ln v + \ln d$

$\therefore \frac{x^2}{y}\sqrt{\frac{y}{x} - \frac{y^3}{x^3}} = d \qquad f\left(\frac{x^2}{y}\sqrt{\frac{y}{x} - \frac{y^3}{x^3}},\ \frac{(x^2-y^2)}{z}\right) = 0$

i. (b) Solve $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y$. **[06]**

Auxillary eqn is

$(D^2 - DD' + D' - 1)z = 0$ put $D = m, D' = 1$

~~$m^2 - m + D(D-D')$~~

$(D^2 - 1) - D'(D-1)z = 0$

$\therefore (D-1)(D - D' + 1)z = 0$

$\therefore z_c = e^{-x}\phi_1(y+x) + e^{x}\phi_2(y)$ — ①

where ϕ_1, ϕ_2 are arbitrary functions.

Particular Integral

$$z_p = \frac{1}{(D^2 - DD' + D' - 1)}\cos(x+2y) + e^y$$

$$\frac{1}{(D^2 - DD' + D' - 1)}\cos(x+2y) = \frac{1}{(-1 + 2 + D' - 1)}\cos(x+2y)$$

$$= \frac{1}{D'}\cos(x+2y)$$

$$= \frac{\sin(x+2y)}{2}$$

$$\frac{1}{(D^2 - DD' + D' - 1)}e^y = e^y \frac{1}{[D^2 - D(D'+1) + D' + 1 - 1]}1$$

$$= e^y \frac{1}{D^2 - DD' - D + D'}1$$

04

$$= \frac{e^y}{D'}\left[1 + \frac{D^2 - DD' - D}{D'}\right]^{-1}1$$

$$= e^y y$$ — xe^y

$$z_p = \frac{\sin(x+2y)}{2} + e^y y \quad \text{— ②}$$

From ① & ② $z = z_c + z_p$

6. (c) Reduce the equation $yr + (x + y)s + xt = 0$ to canonical form and hence find its general solution. [14]

Comparing with $Rr + Ss + Tt = 0$

$R = y \quad S = (x+y) \quad T = x$

$\therefore \; y\lambda^2 + (x+y)\lambda + x = 0$

$$\therefore \; \lambda = \frac{-(x+y) \pm \sqrt{(x+y)^2 - 4xy}}{2(y)}$$

$$\lambda = \frac{-(x+y) \pm (x-y)}{2y} \Rightarrow \frac{-x-y+x-y}{2y}, \; \frac{-x-y-x+y}{2y}$$

$$\lambda = -1, \; \frac{-x}{y}$$

$$\frac{dy}{dx} + \lambda_1 = 0 \Rightarrow \frac{dy}{dx} - 1 = 0 \Rightarrow y - x = u \quad -(1)$$

$$\frac{dy}{dx} + \lambda_2 = 0 \Rightarrow \frac{dy}{dx} - \frac{x}{y} = 0 \Rightarrow y^2 - x^2 = v \quad -(2)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(-1) + \frac{\partial z}{\partial v}(-2x) \quad \Big| \quad \frac{\partial}{\partial x} = -\frac{\partial}{\partial u} - 2x\frac{\partial}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(2y) \quad \Big| \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial z}{\partial u} - 2x\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} - 2\frac{\partial z}{\partial v} - 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} - 2\frac{\partial z}{\partial v} + 2x\frac{\partial^2 z}{\partial u \partial v} + 4x^2\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 4x\frac{\partial^2 z}{\partial u \partial v} + 4x^2\frac{\partial^2 z}{\partial v^2} - 2\frac{\partial z}{\partial v} \quad -(3)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right]$$

$$= -\frac{\partial^2 z}{\partial u^2} - 2x\frac{\partial^2 z}{\partial u \partial v} - 2y\frac{\partial^2 z}{\partial u \partial v} - 4xy\frac{\partial^2 z}{\partial v^2} \quad -④$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right]$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + 2\frac{\partial z}{\partial v} + 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + 2y\frac{\partial^2 z}{\partial u \partial v} + 4y^2\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial z}{\partial v} \quad -⑤$$

$$= \frac{\partial^2 z}{\partial u^2} + 4y\frac{\partial^2 z}{\partial u \partial v} + 4y^2\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial z}{\partial v} \quad -⑤$$

③ × y + (x+y) × ④ + x × ⑤

$$\therefore yr + (x+y)s + xt = \frac{\partial^2 z}{\partial u^2}(0) + \frac{\partial^2 z}{\partial u \partial v}\left[-2(x-y)^2\right]$$

$$+ \frac{\partial^2 z}{\partial v^2}(0) + -2y\frac{\partial z}{\partial v} + 2x\frac{\partial z}{\partial v} = 0$$

$$\therefore -2(x-y)^2\frac{\partial^2 z}{\partial u \partial v} + 2(x-y)\frac{\partial z}{\partial v} = 0$$

$$-\cancel{2}u^2\frac{\partial^2 z}{\partial u \partial v} + \cancel{2}u\frac{\partial z}{\partial v} = 0$$

$$uv\frac{\partial^2 z}{\partial u \partial v} + v\frac{\partial z}{\partial v} = 0 \qquad \text{Put } u = e^X,\ v = e^Y$$

12

$$(DD' + D')z = 0$$

$$D'(D+1)z = 0 \Rightarrow z = \phi(X) + e^{-X}\phi(Y)$$

$$z = \phi(\ln u) + \frac{1}{u}\phi(\ln v)$$

$$z = \phi(y-x) + \frac{1}{y-x}\phi(y^2-x^2) \text{ is reqd soln.}$$

6. (d) A tightly stretched elastic string of length l, with fixed end points $x = 0$ and $x = l$ is initially in the position given by $y = C \sin^3(\pi x/l)$, C being constant. It is released from the position of rest. Find the displacement y (x, t). [18]

$\cancel{f(x)}$ $y(x,0) = c\sin^3\left(\frac{\pi x}{l}\right)$ $\quad \frac{\partial y}{\partial t}(x,0) = 0$

2D wave equation is

$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

$y(x,0) = f(x)$
$y(0,t) = 0$
$y(l,t) = 0$

let $u = X(x)T(t)$

$\therefore X''T = \frac{1}{c^2}T''X \Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = \mu$

case I) $\mu = 0$

$X'' = 0 \Rightarrow X = c_1 + c_2 x$

$X(0) = 0 \Rightarrow c_1 = 0$. $X(l) = 0 \Rightarrow c_2 = 0$

$\therefore y = 0$ Not true

case II) $\mu > 0 = \lambda^2$

$X'' = \lambda^2 X \Rightarrow X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$X(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$X(l) = 0 \Rightarrow c_1\left[e^{\lambda l} - e^{-\lambda l}\right] = 0 \Rightarrow c_1 = 0$, $c_2 = 0$

$y = 0 \Rightarrow$ Not true

case III) $\mu < 0 = -\lambda^2$

$\therefore X'' = -\lambda^2 X \Rightarrow X = c_1 \cos\lambda x + c_2 \sin\lambda x$

$X(0) = 0 \Rightarrow c_1 = 0$

$X(l) = 0 \Rightarrow c_2 \sin\lambda l = 0 \Rightarrow \lambda l = n\pi$
$\lambda = \frac{n\pi}{l}$

$\therefore X = \sum B_n \sin\left(\frac{n\pi x}{l}\right)$

$\frac{T''}{c^2 T} = -\lambda^2 \Rightarrow T'' = -\lambda^2 c^2 T$

$\therefore T = C\cos\lambda ct + D\sin\lambda ct$

$$\therefore y = \sum C_n \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right)$$

$$\therefore y(x,t) = \sum \sin\left(\frac{n\pi x}{l}\right)\left[E_n \cos\left(\frac{n\pi ct}{l}\right) + F_n \sin\left(\frac{n\pi ct}{l}\right)\right]$$

$$y(x,0) = \sum \sin\left(\frac{n\pi x}{l}\right) E_n = c \sin^3\left(\frac{\pi x}{l}\right)$$

$$\frac{\partial y}{\partial t} = \sum \sin\left(\frac{n\pi x}{l}\right)\left[F_n\left(\frac{n\pi c}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)\right] = 0 \Rightarrow F_n = 0$$

$$y(x,0) = \sum E_n \sin\frac{n\pi x}{l} = c\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right]$$

$$\therefore E_1 = \frac{3c}{4} \qquad E_3 = -\frac{c}{4} \qquad E_n = 0 \quad n \neq 1, 3$$

$$\therefore y(x,t) = \frac{3c}{4}\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right) - \frac{c}{4}\sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{3\pi ct}{l}\right)$$

15

7. (a) The bacteria concentration in a reservoir varies as $C=4e^{-2t}+e^{-0.1t}$. Using Newton Raphson method, calculate the time required for the bacteria concentration to be 0.5. [10]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ — Newton Raphson

$$f(t) = 4e^{-2t} + (e)^{-0.1t} - 0.5$$

$$f'(t) = -8e^{-2t} + e^{-0.1t}(-0.1)$$

$$= -8e^{-2t} - 0.10e^{-0.1t}$$

02

$$x_{n+1} = x_n - \frac{(4e^{-2t} + e^{-0.1t})}{(-8e^{-2t} - 10e^{-0.1t})}$$

$$\frac{dx}{dt} = 4e^{-2t} + e^{-0.1t}$$

(b) Solve $20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$ by Gauss Seidal method. **[10]**

$$x = \frac{1}{20}\left[17 - y + 2z\right]$$

$$y = \frac{1}{20}\left[-18 - 3x + z\right]$$

$$z = \frac{1}{20}\left[25 + 3y - 2x\right]$$

Iteration	x	y	z
1.	0	0	0
2.	$\frac{17}{20}$	-1.0275	1.0108
3.	1.0024	-0.9998	0.99977
4.	0.9999	-1.0000	1.0000
5	1	-1	$+1$

$\therefore \boxed{x, y, z = (1, -1, 1)}$

-09-

7. (c) A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below:

h (ft.)	10	11	12	13	14
A (sq. ft.)	950	1070	1200	1350	1530

If t denotes time in minutes, the rate of fall of the surface is given by $\frac{dh}{dt} = -48\sqrt{h/A}$.

Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices. [13]

$\frac{dh}{dt} = -48\sqrt{\frac{h}{A}}$ $\quad dt = \int_{h=14}^{h=10} \frac{-1}{48}\sqrt{\frac{A}{h}}$

h	10	11	12	13	14
A	950	1070	1200	1350	1530
$\frac{dh}{dt}$	-4.9246	-4.8668	-4.8	-4.7102	-4.5915
$1/f(h) =$	-0.2030	-0.2054	-0.2083	-0.2123	-0.24779

By Simpson's $\frac{1}{3}$rd method

$$dt = t = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3) + 2(y_2)\right]$$

$$= \frac{1}{3}\left[-2.50819\right]$$

$$t = -0.83606$$

$$|t| = 0.83606$$ minutes

11

7. (d) (i) Find the decimal equivalent of $(357.32)_8$.

(ii) Draw a flow chart for Runge Kulta Method. [17]

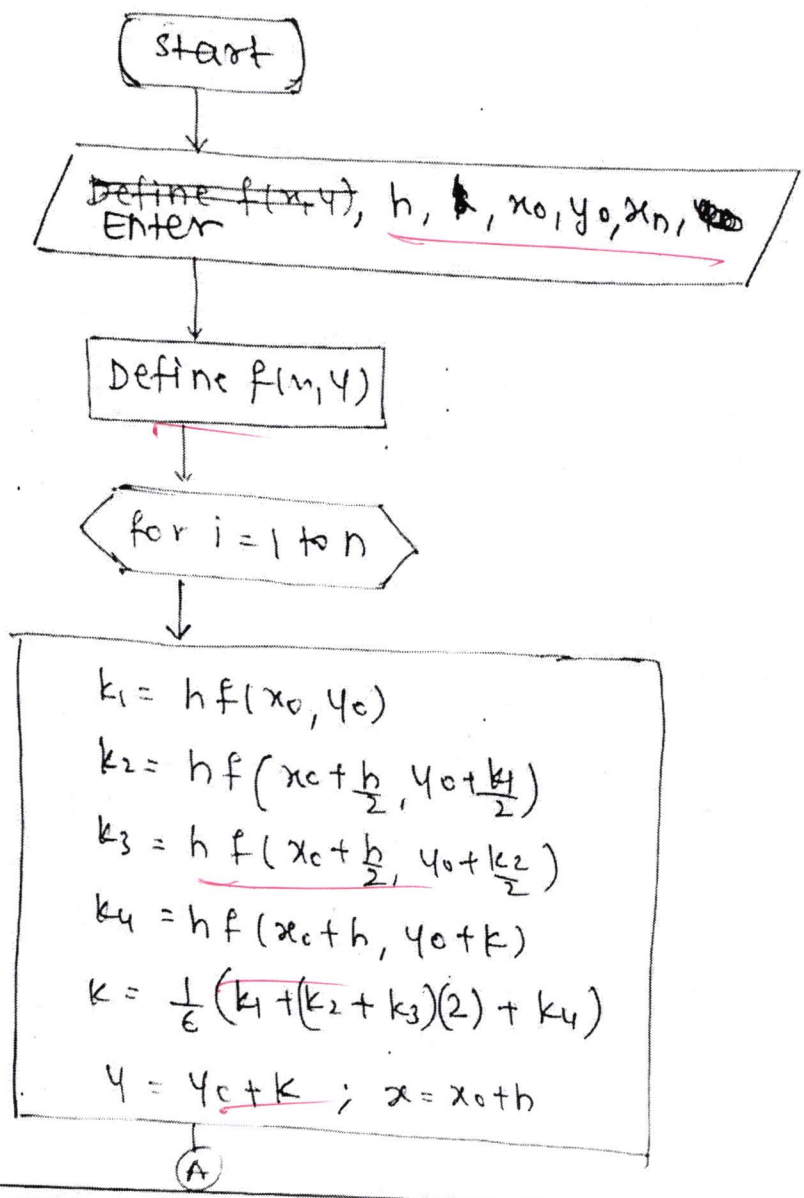
$(357.32)_8$

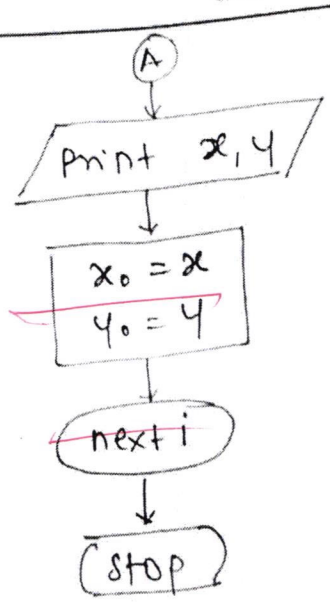
$= 3\times 8^2 + 5\times 8^1 + 7\times 8^0 + 3\times\frac{1}{8} + 2\times\frac{1}{8^2}$

$= 239 + 0.40625$

$(357.32)_8 = (239.40625)_{10}$.

(ii)





12

8. (a) A homogeneous sphere of radius, rotating with angular velocity ω about horizontal diameter is gently placed on a table whose coefficient of friction is μ. show that there will be slipping at the point of contact for a time $(2a\omega/7\mu g)$. and that then the sphere will roll with angular velocity $(2\omega/7)$. [18]