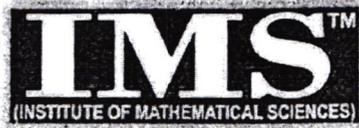


186
250

Date :

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



MAINS TEST SERIES-2020

(OCT. TO JAN.-2020-21)

IAS/IFoS

MATHEMATICS

Under the guidance of K. Venkanna

FULL SYLLABUS (PAPER-I)

DATE : 06-DEC.-2020

Common Test
Test-15 for Batch-I
&
Test-7 for Batch-II

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 43 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name Vinayak Narwade

Roll No.

Test Centre ONLINE

Medium English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

Vinayak

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			9
	(b)			3
	(c)			7
	(d)			8
	(e)			8
2	(a)			8
	(b)			10
	(c)			
	(d)			
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			16
	(b)			14
	(c)			13
	(d)			
5	(a)			9
	(b)			
	(c)			6
	(d)			6
	(e)			8
6	(a)			14
	(b)			6
	(c)			17
	(d)			
7	(a)			10
	(b)			14
	(c)			18
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
Total Marks				

SECTION - A

1. (a) Let $u = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

(i) Find a vector w_1 , different from u and v , so that $\langle \{u, v, w_1\} \rangle = \langle \{u, v\} \rangle$.

(ii) Find a vector w_2 so that $\langle \{u, v, w_2\} \rangle \neq \langle \{u, v\} \rangle$.

[10]

i) If w_1 is a linear combination of u, v then
 $\langle \{u, v, w_1\} \rangle = \langle \{u, v\} \rangle$

This is possible if u, v are linearly independent (LI)

To check LI = $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ $R_2 - 2R_1 \rightarrow R_2$

= $\begin{bmatrix} 1 & 3 & -2 \\ 0 & -8 & 5 \end{bmatrix}$ \Rightarrow They are indeed LI

\therefore let $w_1 = u + v = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

$\therefore \langle \{u, v, w_1\} \rangle = \langle \{u, v\} \rangle$

ii) $\langle \{u, v, w_2\} \rangle \neq \langle \{u, v\} \rangle$ if u, v, w_2 are LI

choosing $w_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \langle \{u, v, w_2\} \rangle \neq \langle \{u, v\} \rangle$

09

1. (b) Let $T: \mathbb{C} \rightarrow M_{2,2}$ be given by

$$T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} a+b & a+b+c \\ a+b+c & a+d \end{bmatrix}. \text{ Find a basis of } R(T). \text{ Is } T \text{ surjective?}$$

[10]

$$T: \mathbb{C} \rightarrow M_{2,2} \quad T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+b & a+b+c \\ a+b+c & a+d \end{bmatrix}$$

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \text{Basis of } R(T) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Please proceed ahead to examine it

03

1. (c) (i) Evaluate $\left(\frac{\tan x}{x}\right)^{1/x^2}$, $(x \rightarrow 0)$

(ii) If $z = (x+y) + (x+y)\phi(y/x)$, prove that

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$$

[10]

c) i) $L = \lim_{n \rightarrow 0} \left(\frac{\tan n}{n}\right)^{1/n^2}$ 1[∞] form

$$\ln L = \lim_{n \rightarrow 0} \frac{1}{n^2} \ln\left(\frac{\tan n}{n}\right) \quad \left(\frac{0}{0}\right)$$

Applying L Hospital's rule

$$\ln L = \lim_{n \rightarrow 0} \frac{\frac{x}{\tan x} \left[\frac{n \sec^2 x - \tan x}{n^2} \right]}{2x}$$

$$= \lim_{n \rightarrow 0} \left[\frac{n \sec^2 x - \tan x}{2n^3} \right] \frac{x}{\tan x}$$

$$= \lim_{n \rightarrow 0} \left[\frac{2x \sec^2 x \tan x + \sec^2 x - \sec^2 x}{6n^2} \right] \times 1$$

$$\ln L = \lim_{n \rightarrow 0} \frac{2 \sec(0)}{6} \frac{\tan n}{n} = \frac{1}{3}$$

$$L = e^{1/3}$$

ii) $z = (n+y) + (n+y)\phi\left(\frac{y}{n}\right)$

$$\frac{\partial z}{\partial x} = 1 + \phi\left(\frac{y}{n}\right) - \frac{(n+y)\phi'\left(\frac{y}{n}\right)}{n^2} \quad \frac{\partial z}{\partial y} = 1 + \phi\left(\frac{y}{n}\right) + (n+y)\phi'\left(\frac{y}{n}\right)\left(\frac{1}{n}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \phi'\left(\frac{y}{n}\right)\left(-\frac{1}{n^2}\right) - \phi''\left(\frac{y}{n}\right)\frac{y(n+y)}{n^3} \quad \frac{\partial^2 z}{\partial y^2} = \phi''\left(\frac{y}{n}\right)\left(\frac{1}{n}\right) + \frac{(n+y)\phi''\left(\frac{y}{n}\right)}{n^2} + \frac{1}{n}\phi'\left(\frac{y}{n}\right)$$

$$\frac{\partial^2 z}{\partial n^2} = \phi\left(\frac{y}{n}\right)\left(-\frac{2y}{n^3}\right) + \frac{y^2(n+y)\phi''\left(\frac{y}{n}\right)}{n^4} - y\left(\frac{1}{n^2} + \frac{2y}{n^3}\right)\phi'\left(\frac{y}{n}\right)$$

$$n\left(\frac{\partial^2 z}{\partial n^2}\right) - \frac{\partial^2 z}{\partial y \partial n} = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial n \partial y}\right) = y\left[\left(\frac{y+1}{n} + \frac{2y}{n^2}\right)\phi' + \phi''\left[\frac{2xy^2}{n^2} + \frac{y^2}{n^3}\right]\right]$$

1. (d) For the function

$$f(x,y) = \begin{cases} \frac{x^2 - x\sqrt{y}}{x^2 + y}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

[10]

Examine the continuity and differentiability.

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - x\sqrt{y}}{x^2 + y}$$

$$(x,y) \rightarrow (0,0) \text{ Along } y = mx^2$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - x^2\sqrt{m}}{x^2 + mx^2}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{1 - \sqrt{m}}{1 + m}$$

⊙ Tends to different limit for diff
m

∴ Function is not continuous and hence
not differentiable @ (0,0), but elsewhere
it is!

1. (e) If the axes are rectangular, find the S.D. between the lines $y = az + b$, $z = \alpha x + \beta$ and $y = a'z + b'$, $z = \alpha'x + \beta'$. Also deduce the condition for the lines to intersect. [10]

$$y = az + b \quad z = \alpha x + \beta \quad \text{--- (1)}$$

Let l, m, n be direction ratios of lines.

$$\therefore \alpha l + n = 0 \quad \Rightarrow \quad \frac{l}{n} = -\frac{1}{\alpha}$$

$$a n - m = 0 \quad \Rightarrow \quad \frac{m}{n} = a$$

$$\therefore \frac{l}{a} = \frac{m}{a} = \frac{n}{\alpha} \quad \alpha l = m = n$$

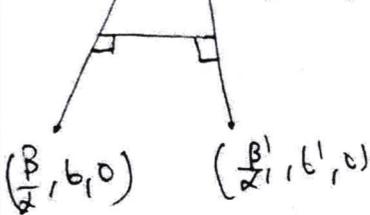
$$\therefore \frac{l}{1} = \frac{m}{a\alpha} = \frac{n}{\alpha}$$

① passes through $(-\frac{\beta}{\alpha}, b, 0)$

$$\therefore \text{①} \equiv \frac{x - \frac{\beta}{\alpha}}{1} = \frac{y - b}{a\alpha} = \frac{z}{\alpha}$$

Similarly ② can be written as $\frac{x - \frac{\beta'}{\alpha'}}{1} = \frac{y - b'}{a'\alpha'} = \frac{z}{\alpha'}$

$(1, \alpha a, \alpha)$ $(1, a'\alpha', \alpha')$



If L, M, N are DRS of SD

$$\therefore L + M\alpha + N\alpha = 0$$

$$L + M\alpha' + N\alpha' = 0$$

$$\frac{L}{\alpha\alpha' - a'\alpha} = \frac{-M}{\alpha' - \alpha} = \frac{N}{a'\alpha' - a\alpha}$$

$$SD = \alpha\alpha'(a - a')\left(\frac{\beta'}{\alpha'} - \frac{\beta}{\alpha}\right) + (\alpha - \alpha')(b' - b) + (a'\alpha' - a\alpha)c$$

$$\sqrt{(\alpha\alpha'(a - a'))^2 + (\alpha - \alpha')^2(b' - b)^2 + (a'\alpha' - a\alpha)^2}$$

$$SD = \alpha\alpha'(a - a')(\alpha\beta' - \alpha'\beta) + (\alpha - \alpha')(b' - b)$$

$$\sqrt{(\alpha\alpha')^2(a - a')^2 + (\alpha - \alpha')^2(b' - b)^2 + (a'\alpha' - a\alpha)^2}$$

2 lines intersect if $SD = 0$.

$$(a - a')(\alpha\beta' - \alpha'\beta) + (\alpha - \alpha')(b' - b) = c$$

2. (a) (i) Suppose that $\{v_1, v_2, v_3, \dots, v_n\}$ is a set of vectors. Prove that $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, \dots, v_n - v_1\}$ is a linearly dependent set.
 (ii) Suppose that $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set in \mathbb{C}^{35} . Prove that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\}$ is a linearly independent set.
 (iii) Find a basis for the subspace W of \mathbb{C}^4 .

[5+5+10=20]

$$W = \left\{ \begin{bmatrix} a+b-2c \\ a+b-2c+d \\ -2a+2b+4c-d \\ b+d \end{bmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$$

i) $v_1 \dots v_n$ are vectors.

To prove let

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + \dots + a_n(v_n - v_1) = 0 \quad \text{--- (1)}$$

$$\therefore a_1 v_1 - a_1 v_2 + a_2 v_2 - a_2 v_3 + \dots + a_n v_n - a_n v_1 = 0$$

Putting $a_1 = a_n = \dots = a_n = 1$ satisfies (1)

$\therefore v_1 - v_2 \dots v_n - v_1$ are LI vectors

ii) v_1, v_2, v_3, v_4 are LI

$$\therefore \text{let } a_1 v_1 + a_2(v_1 + v_2) + a_3(v_1 + v_2 + v_3) + a_4(v_1 + v_2 + v_3 + v_4) = 0$$

$$\therefore v_1(a_1 + a_2 + a_3 + a_4) + v_2(a_2 + a_3 + a_4) + v_3(a_3 + a_4) + v_4(a_4) = 0$$

$\therefore v_i$ are LI

$a_4 = 0$

$a_3 + a_4 = 0 \Rightarrow a_3 = 0$

$a_2 + a_3 + a_4 = 0 \Rightarrow a_2 = 0$

$a_1 \neq 0$

LI

$$W: \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 1 & -2 & 1 \\ -2 & 2 & 4 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\rightarrow \left\{ \begin{matrix} (1 \ 2 \ 0) \\ (0 \ 0 \ 2 \ 1) \\ (0 \ 0 \ 0 \ 1) \end{matrix} \right\}$$

10

4. (a) (i) Define $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $T((z_1, z_2)) = (iz_1, (1+i)z_2 - z_1)$. Let \mathbb{C}^2 have the basis $S = \{(i, 0), (0, 1)\}$. Calculate M_T .

(ii) If A is a non-singular matrix, then show that $\text{adj adj } A = |A|^{n-2} A$.

(iii) Using Cayley-Hamilton theorem, find A^8 , if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

[18]

$$T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$$

$$T(z_1, z_2) = (iz_1, (1+i)z_2 - z_1)$$

$$\text{Basis } S = \{(i, 0), (0, 1)\}$$

$$T(i, 0) = (-1, -i) = i(i, 0) - i(0, 1)$$

$$T(0, 1) = (0, (1+i)) = 0(i, 0) + (1+i)(0, 1)$$

$$M_T = \begin{bmatrix} i & 0 \\ -i & 1+i \end{bmatrix}$$

is matrix representation
of T

$$ii) \quad A \operatorname{adj} A = \operatorname{adj} A A = |A| I$$

$$|A \operatorname{adj} A| = ||A| I|$$

$$|A| |\operatorname{adj} A| = |A|^n$$

$$\therefore |\operatorname{adj} A| = |A|^{n-1} \quad \text{--- (1)}$$

$$\therefore |AB| = |A| |B|$$

$$|kI| = k^n |I|$$

$$\operatorname{adj} A \operatorname{adj}(\operatorname{adj} A) = |\operatorname{adj} A| I$$

$$\therefore |\operatorname{adj} A \operatorname{adj}(\operatorname{adj} A)| = (|\operatorname{adj} A| |I|)$$

$$\therefore |\operatorname{adj} A| |\operatorname{adj}(\operatorname{adj} A)| = |\operatorname{adj} A|^n$$

$$|\operatorname{adj}(\operatorname{adj} A)| = |\operatorname{adj} A|^{n-1}$$

$$\therefore A \operatorname{adj} A \operatorname{adj}(\operatorname{adj} A) = A |A|^{n-1} I \quad \text{--- Premultiply by } A$$

$$\therefore |A| \operatorname{adj}(\operatorname{adj} A) = |A|^{n-1} A$$

$$\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} A \quad \text{--- ??}$$

iii) Cayley Hamilton Theorem - Every square matrix is a root of its characteristic equation

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \quad \begin{matrix} c_{11} = 0 \\ c_{22} = -5 \end{matrix}$$

$$\therefore \lambda^2 - 5 = 0$$

$$\therefore A^2 = 5I \quad \text{--- By Cayley Hamilton Theorem}$$

$$A^2 \cdot A^2 = 5I \cdot 5I$$

$$A^4 = 25I$$

$$A^8 = 625I$$

$$A^8 = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix} \text{ is required answer}$$

16 ✓

4. (b) (i) Let $z = f(t)$, $t = \frac{x+y}{xy}$. Show that $x^2 \frac{\partial z}{\partial x} = y^2 \frac{\partial z}{\partial y}$.

(ii) Evaluate $\iiint z \, dx \, dy \, dz$ over the volume enclosed between the cone $x^2 + y^2 = z^2$ and the sphere $x^2 + y^2 + z^2 = 1$ on positive side of xy -plane. [16]

$$z = f(t) = f\left(\frac{x+y}{xy}\right)$$

$$\begin{aligned} \frac{\partial z}{\partial x} &= f'\left(\frac{x+y}{xy}\right) \left[\frac{xy(1) - (x+y)(y)}{(xy)^2} \right] \\ &= f'\left(\frac{x+y}{xy}\right) \left[\frac{-y^2}{(xy)^2} \right] = f'\left(\frac{x+y}{xy}\right) \left(-\frac{1}{x^2} \right) \quad \text{---} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= f'\left(\frac{x+y}{xy}\right) \left[\frac{xy(1) - (x+y)(x)}{(xy)^2} \right] \\ &= f'\left(\frac{x+y}{xy}\right) \left[-\frac{1}{y^2} \right] \quad \text{---} \end{aligned}$$

$$\therefore x^2 \frac{\partial^2 z}{\partial x^2} = y^2 \frac{\partial^2 z}{\partial y^2} = -f\left(\frac{x+y}{xy}\right) \text{ - Hence proved}$$

$$(ii) \quad x^2 + y^2 = z^2 \quad x^2 + y^2 + z^2 = 1$$

$$I = \iiint \int z \, dx \, dy \, dz$$

$$z = \sqrt{x^2 + y^2}$$

$$\iint_R \left[\frac{z^2}{2} \right]_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dx \, dy$$

$$\frac{1}{2} \iint_R 1 - x^2 - y^2 - x^2 - y^2 \, dx \, dy$$

$$\frac{1}{2} \iint_R 1 - 2x^2 - 2y^2 \, dx \, dy$$

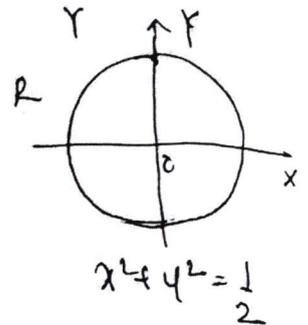
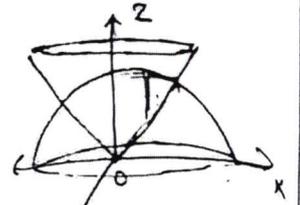
$$\frac{1}{2} \int_0^{2\pi} \int_0^{\sqrt{2}} (1 - 2r^2) r \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{2r^4}{4} \right]_0^{\sqrt{2}} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left[\frac{1}{4} - \frac{1}{2} \left(\frac{1}{4} \right) \right] d\theta$$

$$= \frac{1}{2} \times \frac{1}{8} \times \left[\theta \right]_0^{2\pi}$$

$$I = \frac{\pi}{8}$$



put $x = r \cos \theta$, $y = r \sin \theta$

$r = 0$ to $\frac{1}{\sqrt{2}}$ $\theta = 0$ to 2π

4. (c) CP, CQ are any two conjugate semi-diameters of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $z = c$, CP', CQ' are the conjugate diameters of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $z = -c$ drawn in the same directions as CP and CQ, Prove that the hyperboloid $(2x^2/a^2) + (2y^2/b^2) - (z^2/c^2) = 1$ is generated by either PQ' or P'Q'. [16]

$$\text{Let } P = (a \cos \theta, b \sin \theta, c)$$

$$\therefore Q = (-a \sin \theta, b \cos \theta, c)$$

$$\therefore \underline{CP \perp CQ}$$

$$P' = (a \cos \theta, b \sin \theta, -c)$$

$$Q' = (-a \sin \theta, b \cos \theta, -c)$$

Consider line PQ'

$$\text{Direction ratios} = (a(\cos \theta + \sin \theta), b(\sin \theta - \cos \theta), 2c)$$

$$\therefore PQ' \Rightarrow \frac{x - a \cos \theta}{a(\cos \theta + \sin \theta)} = \frac{y - b \sin \theta}{b(\sin \theta - \cos \theta)} = \frac{z - c}{2c} = \lambda$$

Any point on PQ' is

$$T (a \cos \theta (1 + \lambda) + a \sin \theta \lambda, b \sin \theta (1 + \lambda) - b \cos \theta \lambda, c(1 + 2\lambda))$$

is generated by PQ' if

$$\frac{2x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 1$$

any point lies on it

$$\frac{2x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 2 \left[(\cos \theta (1 + \lambda) + \sin \theta \lambda)^2 + 2 (\sin \theta (1 + \lambda) - \cos \theta \lambda)^2 - (1 + 2\lambda)^2 \right]$$

$$= 2 \left[(1 + \lambda)^2 (\cos^2 \theta + \sin^2 \theta) + 2 \cos \theta \sin \theta (1 + \lambda) - 2 \cos \theta \sin \theta (1 + \lambda) + \lambda^2 (\sin^2 \theta + \cos^2 \theta) \right] - (1 + 2\lambda)^2$$

$$= 2 \left[(1 + 2\lambda)^2 + \lambda^2 \right] - (1 + 2\lambda)^2$$

$$= 2 \left[1 + 2\lambda + 2\lambda^2 \right] - (1 + 4\lambda + 4\lambda^2) = 1$$

Thus any point T on PQ' lies on the hyperbola.

Similarly it can be shown that any point S on $P'Q$ lies on hyperbola.

$\therefore \frac{2x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 1$ is generated by either PQ' or $P'Q$

SECTION - B

5. (a) (i) Solve : $x \cos(y/x) (y dx + x dy) = y \sin(y/x) (x dy - y dx)$
 (ii) Solve $y(x^2y^2 + 2) dx + x(2 - 2x^2y^2) dy = 0$

[10]

$$x \cos\left(\frac{y}{x}\right) (y dx + x dy) = y \sin\left(\frac{y}{x}\right) (x dy - y dx)$$

$$\frac{(y dx + x dy)}{xy} = \frac{\sin(y/x)}{\cos(y/x)} \left[\frac{x dy - y dx}{x^2} \right]$$

$$\frac{d(xy)}{xy} = \frac{\sin(y/x)}{\cos(y/x)} \left[d\left(\frac{y}{x}\right) \right]$$

Integrating both sides

$$\ln(xy) = -\ln \cos(y/x) + C$$

$$\boxed{xy \cos\left(\frac{y}{x}\right) = C'}$$

$$\underline{\underline{C' = e^C}}$$

$$(11) \quad y(x^2y^2+2) dx + x(2-2x^2y^2) dy = 0$$

$$M = y(x^2y^2+2)$$

$$N = x(2-2x^2y^2)$$

$$\frac{\partial M}{\partial y} = (3x^2y^2+2)$$

$$\frac{\partial N}{\partial x} = (2)(1-3x^2y^2)$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is of form $y f_1(x, y) dx + x f_2(x, y) dy$

$$\text{Integrating factor} = \frac{1}{Mx - Ny}$$

$$\therefore Mx - Ny = xy(x^2y^2+2) - x(2-2x^2y^2) \\ = 3(xy)^3$$

$$\therefore \frac{y(x^2y^2+2)}{3(xy)^3} dx + \frac{x(2-2x^2y^2)}{3(xy)^3} dy = 0$$

$$\int_{y=\text{const}} M dx + \int_{\substack{\text{Terms of } N \text{ not} \\ \text{containing } x}} N dy = C$$

$$\therefore \int_{y=\text{const}} \frac{y}{3y^2} \left[\frac{y^2}{x} + \frac{2}{x^3} \right] dx = C$$

$$\frac{1}{3y^2} \left[y^2 \ln x + \frac{2x^{-2}}{-2} \right] = C$$

$$\boxed{\frac{1}{3} \left[\ln x - \frac{1}{xy^2} \right] = C}$$

is required solution

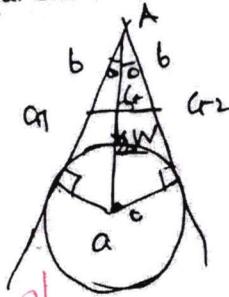
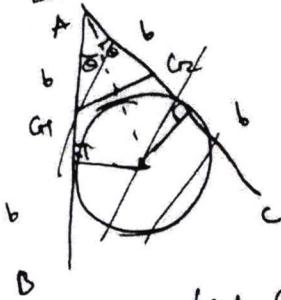
5. (b) Solve $(px^2 + y^2)(px + y) = (p + 1)^2$ by reducing it to Clairaut's form and find its singular solution. [10]

$$(px^2 + y^2)(px + y) = (p + 1)^2$$

$$(px^2 + y^2)(px + y) = (p + 1)^2 \quad \text{--- } y$$

DO
here only

5. (c) Two equal rods, AB and AC, each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a . Show that if 2θ be the angle between them then $b \sin^3 \theta = a \cos \theta$. [10]



Let G be the common CG of rods

Let W be weight of each rod.

By virtual work

$$2W \delta(OG) = 0$$

$$OG = OA - AG$$

$$OG = a \cos \theta - b \cos \theta$$

$$\therefore \delta(OG) = -a \cos \theta \cot \theta + b \sin \theta$$

$$= 2W [b \sin \theta - a \cos \theta \cot \theta]$$

$$b \sin \theta = \frac{a \cos \theta}{\sin^2 \theta}$$

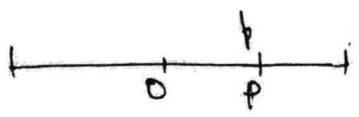
$$b \sin^3 \theta = a \cos \theta$$

Please provide brief description

5. (d) A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P where $OP = b$ with velocity v in the direction OP . prove that the time which elapses before it returns to P is

$$\frac{T}{\pi} \tan^{-1} \left(\frac{vT}{2\pi b} \right).$$

[10]


 $\ddot{x} = -\mu x$ be SHM
 $2\dot{x}\ddot{x} = -\mu x(2\dot{x})$

$$\therefore (\dot{x})^2 = -\mu x^2 + C \quad \text{--- (1)}$$

At $x = b$, $\dot{x} = v$

$$\therefore (v^2) = -\mu b^2 + C$$

$$\therefore \dot{x}^2 = \mu (b^2 - x^2) + v^2$$

$$\dot{x} = \sqrt{\mu (b^2 - x^2) + v^2} \quad \text{--- (2)}$$

$$\frac{dx}{dt} = \sqrt{\mu b^2 + v^2 - \mu x^2}$$

$\Rightarrow \frac{dx}{dt} = 0$ at amplitude
 $\therefore a^2 = \frac{\mu b^2 + v^2}{\mu}$

$$\frac{dx}{\sqrt{b^2 + \frac{v^2}{\mu} - x^2}} = \sqrt{\mu} dt$$

$$\sin^{-1} \left[\frac{x}{\sqrt{b^2 + \frac{v^2}{\mu}}} \right] = \sqrt{\mu} t + C$$

At $x = b$, $t = 0 \Rightarrow \sin^{-1} \left[\frac{b}{\sqrt{b^2 + \frac{v^2}{\mu}}} \right] = C$

$$\therefore \sin^{-1} \left[\frac{x}{\sqrt{b^2 + \frac{v^2}{\mu}}} \right] = \sqrt{\mu} t + \sin^{-1} \left[\frac{b}{\sqrt{b^2 + \frac{v^2}{\mu}}} \right]$$

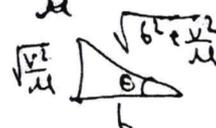
In this case, $\text{time to reach amplitude} = a = \sqrt{\frac{b^2 + \frac{v^2}{\mu}}{\mu}}$

$$\sqrt{\mu} t = \frac{\pi}{2} - \sin^{-1} \left[\frac{b}{\sqrt{b^2 + \frac{v^2}{\mu}}} \right] = \cos^{-1} \frac{b}{\sqrt{b^2 + \frac{v^2}{\mu}}}$$

~~Reqd~~ Reqd time $= 2t = \frac{2}{\sqrt{\mu}} \cos^{-1} \frac{b}{\sqrt{b^2 + \frac{v^2}{\mu}}}$

~~T =~~ $T = \frac{2\pi}{\sqrt{\mu}}$

$$\therefore 2t = \frac{T}{\pi} \tan^{-1} \left[\frac{vT}{2\pi b} \right]$$

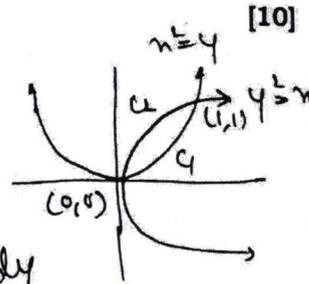


5. (e) Verify Green's theorem in the plane for

$$\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy],$$

where C is the boundary of the region defined by $y = \sqrt{x}$, $y = x^2$.

$$\oint [3x^2 - 8y^2] dx + (4y - 6xy) dy$$



along $C_1 \Rightarrow x^2 = y$

$$\int_0^1 [3x^2 - 8x^4] dx + \int_0^1 [4y - 6y\sqrt{y}] dy$$

$$= \left[\frac{3x^3}{3} - \frac{8x^5}{5} \right]_0^1 + \left[\frac{4y^2}{2} - \frac{6y^{5/2}}{5/2} \right]_0^1$$

$$= \frac{-3}{5} + \left[2 - \frac{12}{5} \right] = -1$$

along $C_2 \Rightarrow y^2 = x$

$$\int_1^0 [3x^2 - 8x] dx + \int_1^0 [4y - 6y^3] dy$$

$$= \left[\frac{3x^3}{3} - 4x^2 \right]_1^0 + \left[2y^2 - \frac{6y^4}{4} \right]_1^0$$

$$= -[+1 - 4] - \left[2(1) - \frac{3}{2} \right] = 3 - \frac{1}{2} = \frac{5}{2}$$

$$I = \int -1 + \frac{5}{2} = \frac{3}{2} \quad \text{--- (1)}$$

By Green's Theorem $\Rightarrow \oint F \cdot dr = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$ $F = \vec{M}\vec{i} + \vec{N}\vec{j}$

$$\therefore I = \int_0^1 \int_{x^2}^{\sqrt{x}} (-6y + 16x) dx dy = \int_0^1 \int_{x^2}^{\sqrt{x}} 5 \left[\frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx dy$$

$$= \int_0^1 5 \left[\frac{x}{2} - \frac{x^5}{5} \right] dx$$

$$= 5 \left[\frac{3}{5 \times 2} \right] = \frac{3}{2} \quad \text{--- (2)}$$

From (1) and (2) Green's Theorem is verified

6. (a) (i) Evaluate $L^{-1}\{e^{4-3s} / (s+4)^{5/2}\}$
 (ii) By using Laplace transform solve $(D^2 + m^2)x = a \sin nt$, $t > 0$ where x, Dx equal to x_0 and x_1 , when $t = 0$, $n \neq m$. [5+13=18]

$$i) f(p) = \frac{e^{4-3p}}{(p+4)^{5/2}} = \frac{e^4 e^{-3p}}{(p+4)^{5/2}}$$

By 1st Laplace transform theorem

$$\begin{aligned} \& L^{-1} e^{-ap} f(p) &= F(t-a) & t > a \\ & &= 0 & t \leq a \end{aligned}$$

$$g(p) = \frac{e^4}{(p+4)^{5/2}}$$

$$L^{-1} g(p) = \frac{e^4 e^{-4t} t^{3/2}}{\Gamma(5/2)} = F(t)$$

$$\begin{aligned} \therefore L^{-1} f(p) &= \frac{e^4 e^{-4(t-3)} (t-3)^{3/2}}{\Gamma(5/2)} & t > 3 \\ &= 0 & t < 3. \end{aligned}$$

$$ii) (D^2 + m^2)x = a \sin nt$$

$$\begin{aligned} x &= x_0 & @ t=0 \\ Dx &= x_1 \end{aligned}$$

Taking Laplace on both sides

$$L(D^2 x + m^2 x) = L(a \sin nt)$$

$$\boxed{L(y^n) = p^n L(y_0) - p^{n-1} y_0' - \dots - y_0^{(n-1)}}$$

$$\therefore p^2 L(x) - p x_0 - x_1 + m^2 L(x) = \frac{an}{p^2 + n^2}$$

$$\therefore (p^2 + m^2) L(x) - p x_0 - x_1 = \frac{an}{p^2 + n^2}$$

$$L(x) = \frac{an}{(p^2 + n^2)(p^2 + m^2)} + \frac{p x_0}{p^2 + m^2} + \frac{x_1}{p^2 + m^2}$$

equal
=18]

$$L(x) = \frac{an}{(m^2-n^2)} \left[\frac{1}{p^2+n^2} - \frac{1}{p^2+m^2} \right] + \frac{px_0}{p^2+m^2} + \frac{xy}{m} \frac{m}{p^2+m^2}$$

Taking Laplace inverse

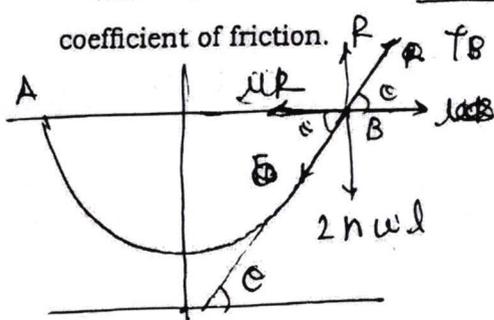
$$x = \frac{an}{m^2-n^2} \left[\frac{\sin nt}{n} - \frac{\sin mt}{m} \right] + x_0 \cos mt + \frac{xy}{m} \sin mt$$

$$x = \sin nt \left[\frac{a}{m^2-n^2} \right] + \frac{\sin mt}{m} \left[xy - \frac{an}{m^2-n^2} \right] + x_0 \cos mt$$

is required solution

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6. (b) A heavy chain, of length $2l$, has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be n times the weight of the chain, show that its greatest possible distance from A is $\frac{2l}{\lambda} \log \left\{ \lambda + \sqrt{1+\lambda^2} \right\}$, where $1/\lambda = \mu (2n-1)$ and μ is the



coefficient of friction.

$$\frac{1}{\lambda} = \mu(2n+1) \quad [14]$$

Let w be weight per unit length of chain

$$\therefore W = 2wl \Rightarrow W \text{ is total weight}$$

$$\therefore \text{weight of ring} = 2nw$$

Let T_B be the tension in wire at B.

R be the reaction at B and θ be $4e9$ tangent at B

$$\therefore T_B \cos \theta = \mu R \quad \text{--- (1)}$$

$$\therefore T_B \sin \theta + 2n\omega l = R$$

$$\therefore T_B \sin \theta + 2n\omega l = \omega l (2n+1) = R \quad \text{--- (2)}$$

$$\therefore T_B \sin \theta = \omega l$$

Also distance ~~is~~ $AB = 2n = 2c \ln [\sec \theta + \tan \theta]$ --- (2)

To prove: $2n = \frac{2l}{\lambda} \ln [d + \sqrt{1+d^2}]$

∴ From (1) and (2)

$$T_B \cos \theta = \mu \omega l (2n+1)$$

$$\mu c = \mu \omega l (2n+1)$$

$$c = \omega l (2n+1)$$

$$c = \frac{\omega l}{\lambda}$$

is characteristic
& constant

~~Now ??~~

$$\frac{T_B \sin \theta}{T_B \cos \theta} = \frac{\omega l}{\mu \omega l (2n+1)}$$

$$= \frac{1}{\mu (2n+1)} = d = \tan \theta$$



$$\therefore 2n = \frac{2l}{\lambda} \ln [\sqrt{1+d^2} + d] \quad \text{--- from (2)}$$

Hence proved

06

6. (c) (i) Find the curvature K , and the torsion τ for the space curve $x = t - t^3/3$, $y = t^2$, $z = t + t^3/3$.

(ii) If $A = 5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k}$ and $B = \sin t\mathbf{i} - \cos t\mathbf{j}$, find $\frac{d}{dt}(A \times B)$, $\frac{d}{dt}(A \times A)$ and $\frac{d}{dt}(A \times A)$

$$x = t - \frac{t^3}{3} \quad y = t^2 \quad z = t + \frac{t^3}{3} \quad [18]$$

$$r = \left(t - \frac{t^3}{3}\right)\mathbf{i} + t^2\mathbf{j} + \left(t + \frac{t^3}{3}\right)\mathbf{k}$$

$$\frac{dr}{dt} = \left(1 - \frac{3t^2}{3}\right)\mathbf{i} + 2t\mathbf{j} + \left(1 + \frac{3t^2}{3}\right)\mathbf{k}$$

$$\frac{d^2r}{dt^2} = (-2t)\mathbf{i} + 2\mathbf{j} + (2t)\mathbf{k}$$

$$\frac{d^3r}{dt^3} = (-2)\mathbf{i} + 0\mathbf{j} + (2)\mathbf{k}$$

$$\left[\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1-t^2 & 2t & 1+t^2 \\ -2t & 2 & 2t \end{vmatrix}$$

$$= \mathbf{i}(2t^2 - 2) - \mathbf{j}(2t - 2t^3 + 2t + 2t^3) + \mathbf{k}(2 - 2t^2 + 4t^2)$$

$$\frac{dr}{dt} \times \frac{d^2r}{dt^2} = \mathbf{i}(2t^2 - 2) - \mathbf{j}(4t) + \mathbf{k}(2 + 2t^2)$$

$$\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right| = \sqrt{4(t^2 - 1)^2 + 16t^2 + 4(1 + t^2)^2}$$

$$= 2\sqrt{2t^4 + 2 + 4t^2}$$

$$\left(\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right) = 2\sqrt{2}(1 + t^2)$$

$$\left[\frac{dr}{dt} \times \frac{d^2r}{dt^2} \cdot \frac{d^3r}{dt^3}\right] = \begin{vmatrix} 1-t^2 & 2t & 1+t^2 \\ -2t & 2 & 2t \\ -2 & 0 & 2 \end{vmatrix}$$

$$= (1-t^2)(4) - (2t)(0) + (1+t^2)(4)$$

$$= 8$$

$$k = \frac{\left| \frac{dr}{dt} \times \frac{dr}{dt^2} \right|}{\left| \frac{dr}{dt} \right|^3}$$

$$k = \frac{2\sqrt{2}(1+t^2)}{[\sqrt{2}(1+t^2)]^3}$$

$$k = \frac{2\sqrt{2}}{2\sqrt{2}(t^2+1)^2}$$

$$k = \frac{1}{(1+t^2)^2}$$

$$z = \frac{\left[\frac{dr}{dt} \frac{dt}{dt^2} \frac{d^2B}{dt^3} \right]}{\left| \frac{dr}{dt} \times \frac{dr}{dt^2} \right|^2}$$

$$z = \frac{8}{8(1+t^2)^2}$$

$$z = \frac{1}{(1+t^2)^2}$$

ii) $A = 5t^2 \bar{i} + t \bar{j} - t^3 \bar{k}$ $B = \sin t \bar{i} - \cos t \bar{j}$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$\bar{A} \times \bar{B} = \bar{i}(-t^3 \cos t) - \bar{j}(t^3 \sin t) + \bar{k}(-5t^2 \cos t - t \sin t)$$

$$\frac{d}{dt}(\bar{A} \times \bar{B}) = \bar{i}(-3t^2 \cos t + t^3 \sin t) - \bar{j}(3t^2 \sin t + t^3 \cos t) + \bar{k}(5t^2 \sin t - 1 + t \cos t - \sin t)$$

$$\bar{A} \times \bar{A} = 0$$

$$\therefore \frac{d}{dt}(\bar{A} \times \bar{A}) = 0$$

print error

$\frac{d}{dt}(\bar{A} \times \bar{B})$ already found

-17-

7. (a) Solve $(x^2 D^2 - xD + 1)y = (\log x \sin \log x + 1)/x$.

[15]

$$(x^2 D^2 - xD + 1)y = [(\ln x) \sin(\ln x) + 1]/x$$

put $x = e^z \quad \therefore \ln x = z$

By Euler's method $xD = D'$ $D' = \frac{d}{dz}$
 $x^2 D^2 = D'(D'-1)$

$$\therefore (D'(D'-1) - D' + 1)y = (z \sin z + 1)e^{-z}$$

$$\therefore (D'^2 - 2D' + 1)y = e^{-z} z \sin z + e^{-z}$$

Auxiliary equation

$$(D'^2 - 2D' + 1)y = 0$$

$$(D' - 1)^2 y = 0 \quad \Rightarrow \quad D' = 1, 1 \text{ twice}$$

$$\therefore y_c = (C_1 + C_2 z) e^z \quad \text{--- ①}$$

Particular integral

$$y = \frac{1}{(D'-1)^2} e^{-z} (z \sin z + 1)$$

$$y = e^{-z} \frac{1}{(D'-1-1)^2} (z \sin z + 1)$$

$$= e^{-z} \left[\frac{1}{(D'-2)^2} (z \sin z + 1) \right]$$

$$= e^{-z} \int$$

$$\frac{1}{(D'-2)^2} 1 = \frac{1}{+4} \left[1 - \frac{D'}{2} \right]^{-2} = \frac{1}{4}$$

$$\frac{1}{(D'-2)^2} z \sin z = \frac{1}{(D'-2)} z^2 \int z^{(1-2)} z \sin z$$

$$\therefore \frac{1}{(D'-2)}$$

$$\int f(x) = x^a \int x^{1-a} f(x) dx$$

$$= \frac{1}{D^2} (-z^2 \cos z) = -z^2 \int z \cos z$$

$$= -z^2 [z \sin z + \cos z]$$

$$= -z^3 \sin z + \cos z - z^2 \cos z$$

$$y_p = e^{-z} \left[\frac{1}{4} - z^3 \sin z - z^2 \cos z \right]$$

$$y_p = e^{-x} \left[\frac{1}{4} - (\ln x)^3 \sin(\ln x) - (\ln x)^2 \cos(\ln x) \right]$$

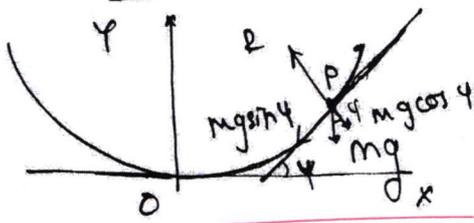
$$y_c = (C_1 + C_2 \ln x) x$$

$$\therefore y = y_p + y_c \text{ is required integral}$$

Please refer key

7. (b) A particle starts from rest at the cusp of a smooth cycloid whose axis is vertical and vertex downwards. Prove that when it has fallen through half the distance measured along the arc to the vertex, two-thirds of the time of descent will have elapsed.

[17]



Let P be position of particle at any time t

The forces acting on particle are as shown.

$$\frac{mv^2}{r} = R - mg \cos \phi$$

$$-m \frac{d^2 s}{dt^2} = mg \sin \phi$$

-ve sign because s is decreasing

$$s = 4a \sin \phi$$

$$\frac{ds}{d\phi} = r = 4a \cos \phi$$

$$\therefore \frac{mv^2}{r} = \rho - mg \cos \phi$$

$$\frac{ds}{dt} = -g \sin \phi$$

$$s = 4a \sin \phi$$

$$\therefore \sin \phi = \frac{s}{4a}$$

$$2 \frac{ds}{dt} \frac{d^2s}{dt^2} = -\frac{g}{4a} s \times 2 \frac{ds}{dt}$$

$$\therefore \left(\frac{ds}{dt} \right)^2 = -\frac{g}{4a} s^2 + C$$

$$\textcircled{a} \quad s = 4a \quad \frac{ds}{dt} = 0 \quad \Rightarrow \text{wsp.}$$

$$\therefore \left(\frac{ds}{dt} \right)^2 = -\frac{g}{4a} s^2 + g(4a)$$

$$\frac{ds}{dt} = -\sqrt{\frac{g}{4a} [(16a^2) - s^2]}$$

-ve sign
 $\therefore s$ decreases
with $\uparrow t$

$$\int_{s=4a}^{s=2a} \frac{ds}{\sqrt{16a^2 - s^2}} = \int_{t=0}^{t=T} -\sqrt{\frac{g}{4a}} dt \quad \textcircled{1}$$

$t=0$

$$\left\{ \sin^{-1} \left[\frac{s}{4a} \right] \right\}_{4a}^{2a} = -\sqrt{\frac{g}{4a}} T$$

$$\therefore \frac{\pi}{6} - \frac{\pi}{2} = -\sqrt{\frac{g}{4a}} T \Rightarrow T = \frac{\pi}{3} \sqrt{\frac{4a}{g}} \quad \textcircled{2}$$

$$\int_{4a}^0 \frac{ds}{\sqrt{16a^2 - s^2}} = \int_0^{t'} -\sqrt{\frac{g}{4a}} dt$$

$$\sin^{-1} \left[\frac{s}{4a} \right]_{4a}^0 = -\sqrt{\frac{g}{4a}} t' \Rightarrow t' = \frac{\pi}{2} \sqrt{\frac{4a}{g}}$$

$$\therefore t' = \frac{3}{2} T \Rightarrow T = \frac{2}{3} t'$$

$\therefore \frac{2}{3}$ hrs of time will elapse in travelling half the distance

7. (c) (i) Find the values of the constants a, b, c so that the directional derivative of $\phi = ax^2 + by^2 + cz^2$ at $(1, 1, 2)$ has a maximum magnitude 4 in the direction parallel to y -axis.
- (ii) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.
- (iii) Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$, where
- $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy -plane. [4+4+12=20]

$$\phi = ax^2 + by^2 + cz^2 \quad @ (1, 1, 2)$$

$$\nabla \phi = 2ax\mathbf{i} + 2by\mathbf{j} + 2cz\mathbf{k}$$

$\nabla \phi \cdot \mathbf{j}$ has maximum magnitude along y axis if it coincides along y axis

$$(\nabla \phi)_{(1,1,2)} = 2a\mathbf{i} + 2b\mathbf{j} + 4c\mathbf{k}$$

coincides with y axis if $a=0, c=0$

$$\nabla \phi \cdot \mathbf{j} = 2b = 4 \Rightarrow b = 2$$

$$\boxed{a=0, c=0, b=2}$$

(ii) Let both surfaces = Let both their normals

Let n_1 and n_2 be normals of $\phi_1 \Rightarrow x^2 + y^2 + z^2 = 9$

and $\phi_2 \Rightarrow z = x^2 + y^2 - 3$ @ $(2, -1, 2)$

$$\begin{aligned} n_1 = \nabla \phi_1 &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \\ &= 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \end{aligned}$$

$$\begin{aligned} n_2 = \nabla \phi_2 &= 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \\ &= 4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \end{aligned}$$

If θ be the angle between them

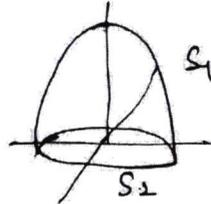
$$\cos \theta = \frac{n_1 \cdot n_2}{|n_1| |n_2|} = \frac{16 + 4 - 4}{\sqrt{36} \sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1} \left(\frac{8}{3\sqrt{21}} \right)$$

(ii) $\vec{F} = (x^2 + y^2 - 4)\vec{i} + 3xy\vec{j} + (2xz + z^2)\vec{k}$

By divergence theorem

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds = \iiint_V \text{div}(\text{curl } \vec{F}) \cdot \vec{n} \, ds$$



$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n} \, ds + \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} \, ds = 0$$

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n} \, ds = - \iint_{S_2} (\nabla \times \vec{F}) \cdot (-\vec{k}) \, ds$$

∴ normal
of S_2
= $-\vec{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 - 4 & 3xy & 2xz + z^2 \end{vmatrix}$$

$$= -\vec{j}(2z) + \vec{k}(3y - 1)$$

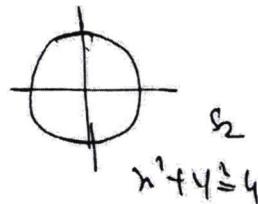
$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n} \, ds = \iint_{S_2} (3y - 1) \frac{dxdy}{|\vec{n} \cdot \vec{k}|} \text{ eliminate } z$$

$$= \iint_{S_2} (3y - 1) \, dxdy$$

Put $x = r \cos \theta$ $y = r \sin \theta$

$$dxdy = r \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^2 (3r \sin \theta - 1) r \, dr \, d\theta$$



$$= \int_0^{2\pi} \int_0^2 (3r^2 \sin \theta - r) dr d\theta$$

$$= \int_0^{2\pi} \left[3r^3 \sin \theta - \frac{r^2}{2} \right]_0^2 d\theta$$

$$= \int_0^{2\pi} [8 \sin \theta - 2] d\theta$$

$$= 8[\cos \theta]_0^{2\pi} - 2[\theta]_0^{2\pi}$$

$$= 8[-1 + 1] - 2(2\pi)$$

$$\iint \nabla \times \vec{F} \cdot \vec{n} dy = \underline{\underline{-4\pi}}$$

8. (a) (i) Find the orthogonal trajectories of