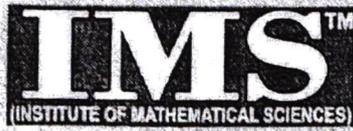


188
252

Date : 4/12/2020

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



MAINS TEST SERIES-2020
(OCT. TO JAN.-2020-21)

IAS/IFoS
MATHEMATICS
Under the guidance of K. Venkanna

Common Test
Test-14 for Batch-I
&
Test-6 for Batch-II

FULL SYLLABUS (PAPER-II)

DATE : 29-NOV-2020

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 40 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name Vinayak Narwade

Roll No.

Test Centre ONLINE

Medium English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate
Vinayak

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

**DO NOT WRITE ON
THIS SPACE**

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			7
	(b)			7
	(c)			8
	(d)			8
	(e)			8
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			14
	(b)			14
	(c)			14
	(d)			
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			8
	(b)			4
	(c)			7
	(d)			9
	(e)			8
6	(a)			15
	(b)			
	(c)			
	(d)			
7	(a)			14
	(b)			15
	(c)			10
	(d)			
8	(a)			8
	(b)			12
	(c)			13
	(d)			
Total Marks				

38
42
36
15
39
33

**DO NOT WRITE ON
THIS SPACE**

SECTION - A

1. (a) Let G be an infinite cyclic group. Prove that e is the only element in G of finite order. [10]

G is infinite cyclic group.

$$\underline{O(e) = 1}$$

Let $a \in G$ $a \neq e$

Suppose $O(a) = m \Rightarrow$ finite

$\therefore H = \{a, a^2, \dots, a^m = e\}$ is a subgroup of G

$$\underline{O(H) = m}$$

\therefore By Lagrange's formula $O(H) \mid O(G)$
 $m \mid \infty \Rightarrow$ not possible

\therefore Our assumption is false

$$\underline{O(a) = \infty} \quad \neq \underline{a \neq e.}$$

07

1. (b) Let R be a commutative ring. Prove that an ideal P of R is a prime ideal of R if and only if $\frac{R}{P}$ is an integral domain. [10]

P is prime ideal $\Leftrightarrow \frac{R}{P}$ is Integral domain

1) P is prime $\Rightarrow \frac{R}{P}$ is ID enough to prove non zero divisors ~~integers~~ exist

$\therefore P$ is prime ideal $\Rightarrow P$ is maximal

\rightarrow ~~Q.E.D.~~ $P \subset (U) \subset R$

Proof $U \subset P \Rightarrow \exists u \in U$ st $u \in R \setminus (U)$

$\therefore R \not\subset P + U \neq P$ $P = UV$ $\therefore P \in (U)$

$\therefore \cancel{P} \subseteq \frac{u \in P}{U \subset P}$ or $\frac{u \in P}{U \in P}$ or $PQ = U$

\therefore u is unit $\therefore U = R$

$\therefore mn \notin P \Rightarrow P + mn \neq P$ and $m \in P, n \in P$.

$\therefore (P+m)(P+n) \neq P \Rightarrow$ NO zero divisors

2) $\frac{R}{P}$ is ID $\Rightarrow P$ is prime

NO zero divisors exist \rightarrow \textcircled{a}

$\therefore mn \in P$

$\therefore P + mn = P$

$\therefore (P+m)(P+n) = P$

\therefore either $P+m = P$ or $P+n = P \Rightarrow P+m$

$\therefore m \in P$ or $n \in P$

$\therefore P$ is prime.

1. (c) Show that $\int_0^t \sin x dx = 1 - \cos t$, by using Riemann integral.

[10]

$$\int_0^t \sin x dx \quad \sin x \text{ is continuous } \forall [0, t]$$

\therefore It is Riemann integrable

$$P = \left[0, \frac{t}{n}, \frac{2t}{n}, \dots, \frac{nt}{n} \right]$$

$$I_r = \left[\frac{(r-1)t}{n}, \frac{rt}{n} \right] \quad r = 1 \text{ to } n. \quad \Delta r = \frac{t}{n}$$

$$f(\xi_r) = \sin\left(\frac{rt}{n}\right) \quad \text{assume } \xi_r = \frac{rt}{n}$$

$$\sum_{r=1}^n f(\xi_r) \Delta r = \left(\sin\left(\frac{t}{n}\right) + \sin\left(\frac{2t}{n}\right) + \dots + \sin\left(\frac{nt}{n}\right) \right) \frac{t}{n}$$

$$\lim_{n \rightarrow \infty} \left\{ \sin\left(\frac{t}{n}\right) + \sin\left(\frac{t}{n} + \frac{t}{n}\right) + \dots + \sin\left(\frac{t}{n} + (n-1)\frac{t}{n}\right) \right\} \frac{t}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{\sin\left[\frac{t}{n} + \frac{(n-1)t}{2n}\right] \sin\left[\frac{nt}{2n}\right] \times \frac{t}{n}}{\sin\left(\frac{t}{2n}\right)}$$

$$\sin \alpha + \sin(\alpha + \beta) + \dots + \sin(\alpha + (n-1)\beta) = \frac{\sin\left(\alpha + \frac{(n-1)\beta}{2}\right) \sin\left(\frac{n\beta}{2}\right)}{\sin \frac{\beta}{2}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sin\left[\frac{t}{2n} + \frac{(n-1)t}{2}\right] \sin\left[\frac{nt}{2}\right] \times \frac{t}{n} \times 2}{\sin\left(\frac{t}{2n}\right)}$$

$$= 2 \sin^2\left(\frac{t}{2}\right) \times \lim_{n \rightarrow \infty} \frac{t/2n}{\sin(t/2n)}$$

$$I = (1 - \cos t)$$

$$\therefore \lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$$

$$\int_0^t \sin x dx = 1 - \cos t$$

1. (d) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals. [10]

(i) $\int_{|z-2|=2} \frac{\log(z+1)}{z-3} dz$ (ii) $\int_{|z|=5} \frac{z+5}{z^2-3z-4} dz$

$$\int_{|z-2|=2} \frac{\ln(z+1)}{z-3}$$

$z=3$ lies in the region

$$\text{Residue of } z=3 \Rightarrow a_{-1} = \lim_{z \rightarrow 3} \frac{\ln(z+1)}{(z-3)}$$

\therefore By

$$= \ln(4)$$

$$\therefore \int_{|z-2|=2} \frac{\ln(z+1)}{z-3} = 2\pi i (\ln 4)$$

\therefore By Cauchy's residue formula

$$\int f(z) = 2\pi i (\text{sum of residues})$$

11) $I = \int_{|z|=5} \frac{z+5}{(z-4)(z+1)} dz$

both poles $z=4$ and $z=-1$ lies within C

$$\begin{aligned} \therefore a_{-1} &= \lim_{z \rightarrow 4} \frac{(z+5)(z+1)}{(z-4)(z+1)} \\ &= \frac{9}{5} \end{aligned}$$

$$b_{-1} = \lim_{z \rightarrow -1} \frac{(z+5)(z+1)}{(z-4)(z+1)}$$

$$= \frac{4}{5}$$

$$\therefore I = 2\pi i \left[\frac{9}{5} - \frac{4}{5} \right]$$

$$= 2\pi i [1]$$

By Cauchy's residue formula

1. (e) Find an optimal solution to the following L.P.P. by computing all basic solutions and then finding one that maximizes the objective function :

$$2x_1 + 3x_2 - x_3 + 4x_4 = 8, \quad x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$

$$x_1, x_2, x_3, x_4 \geq 0, \quad \text{Max. } Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$$

[10]

$$\text{Max } Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$$

$$2x_1 + 3x_2 - x_3 + 4x_4 = 8$$

$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$

$$x_i > 0$$

S.No	Basis	Non basic	Feasible	Z	Optimal
1.	$x_1 = 1$ $x_2 = 2$	$x_3 = 0$ $x_4 = 0$	Yes	8	NO
2.	$x_1 = 45/13$ $x_3 = -14/13$	$x_2 = 0$ $x_4 = 0$	NO	21/13 3	
3.	$x_1 = 24/9$ $x_4 = 7/9$	$x_2 = 0$ $x_3 = 0$	Yes	$\frac{31}{3} = 10.33$	NO
4.	$x_2 = 45/16$ $x_3 = 7/16$	$x_1 = 0$ $x_4 = 0$	Yes	10.1875	NO
5.	$x_2 = 44/13$ $x_4 = -7/13$	$x_1 = 0$ $x_3 = 0$	NO		
6.	$x_3 = 44/17$ $x_4 = 45/17$	$x_1 = 0$ $x_2 = 0$	Yes	28.88	Yes

08

max

3. (a) Let R be the set of all real valued continuous functions on $[0, 1]$. Show that R is a commutative ring with respect to point-wise addition and point-wise multiplication. Is R an integral domain?

$$R = \{ f: [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous} \}$$

defined by

$$(f+g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)$$

To prove R is commutative ring

1) $(R, +)$ is abelian group.

1) Closure - let $f, g, h \in R$

$$(f+g)(x) = f(x) + g(x) \in \mathbb{R} \quad \because f(x), g(x) \in \mathbb{R} \text{ (real values)}$$

Also $f+g$ is continuous \therefore closure satisfied

2) Associative -

$$\begin{aligned} [f+(g+h)](x) &= f(x) + (g+h)(x) \\ &= f(x) + g(x) + h(x) \\ &= (f+g)(x) + h(x) \end{aligned}$$

$$= [(f+g)+h](x) \quad \text{--- satisfied}$$

Existence of identity

3) Right inverse

$$f(x) \in \mathbb{R} \Rightarrow \exists \theta(x) \in \mathbb{R} \text{ st } \theta(x) = 0 \text{ continuous}$$

$$\therefore f(x) + \theta(x) = f(x) \quad \text{--- satisfies identity}$$

Existence of right inverse

$$f \in R \Rightarrow -f \in R \text{ st } -f(x) = -f(x)$$

$$f(x) - f(x) = 0 = \theta(x) \quad \text{--- satisfies right inverse}$$

5.) Abelian / Commutativity

$$\begin{aligned}(f+g)(u) &= f(u) + g(u) \\ &= g(u) + f(u) \\ &= (g+f)(u) \quad \text{---}\end{aligned}$$

\therefore real values

$\therefore (R, +)$ is abelian

To prove (R, \cdot) satisfies closure, commutative,
and associative

6.) Closure

$$(fg)(u) = f(u)g(u) \in \mathbb{R} \text{ real value}$$

also continuous $\therefore f$ & g continuous

$\therefore fg \in \mathbb{R}$

7.) Associative

$$\begin{aligned}[f(gh)](u) &= f(u)(gh)(u) \\ &= f(u)g(u)h(u) \\ &= f(u)(fg)(u)h(u) \\ &= [(fg)h](u) \text{ satisfied.}\end{aligned}$$

8.) Commutative

$$(fg)(u) = f(u)g(u) = g(u)f(u) = (gf)(u) \text{ satisfied}$$

9.) Distributive laws

$$\begin{aligned}a) f[g+h](u) &= f(u)(g+h)(u) = f(u)[g(u)+h(u)] \\ &= f(u)g(u) + f(u)h(u) \\ &= (fg + fh)(u) \quad \text{--- satisfied}\end{aligned}$$

$$b) (f+gh)(u) = f(u) + gh(u) = f(u) + g(u)h(u)$$

$$= (f+g)(u) \times (h)(u)$$

satisfied $\therefore (R, +, \cdot)$ is commutative ring

Integral domain

$$\text{let } f(x) = x - \frac{1}{2}$$

$$= 0$$

$$\frac{1}{2} \leq x \leq 1$$

$$0 \leq x \leq \frac{1}{2}$$

$$g(x) = 0$$

$$= x - \frac{1}{2}$$

$$\frac{1}{2} \leq x \leq 1$$

$$0 \leq x \leq \frac{1}{2}$$

$f, g \in R \Rightarrow$ continuous

$$\therefore (fg)(x) = 0 \quad 0 \leq x \leq \frac{1}{2}$$

but neither $f, g = 0$

\therefore zero divisors exist \Rightarrow Not integral domain

3. (b) Show that $\int_2^{\infty} \frac{\cos x}{\log x} dx$ is conditionally convergent.

[16]

$$I = \int_2^{\infty} \frac{\cos^n x}{\log x} dx$$

$$f(x) = \frac{\cos^n x}{\log x}$$

Discontinuity at ∞ $\lim_{n \rightarrow \infty} \frac{\cos^n x}{\log x} = 0 \Rightarrow$ proper integral.

or let $f(x) = \cos x$ $g(x) = \frac{1}{\log x}$

$$\therefore \int_2^{\infty} f(x) dx = (\sin x) \Big|_2^{\infty} \leq 2$$

bounded

$g(x)$ is monotonically decreasing and converges to 0.

\therefore By ~~Abels~~ test

$$\int_2^{\infty} \frac{\cos n}{\log n} \, dn \text{ is } \underline{\text{convergent.}}$$

consider absolute convergence

$$\left| \int_2^{\infty} \frac{\cos n}{\log n} \, dn \right| \leq \int_2^{\infty} \frac{1}{\log n} \, dn$$

$$f(n) = 1 \quad g(n) = \frac{1}{\log n}$$

$$\int_2^{\infty} f(n) \, dn = \infty \Rightarrow \text{Not } \underline{\text{convergent}}$$

\therefore Not ~~absolutely~~ convergent

only ~~conditional~~ convergence

-14-

3. (c) Solve the following linear programming problem by simplex method.
 Max. $z = -2x_1 - x_2$, subject to $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \leq 4$, and $x_1, x_2 \geq 0$.

Max $z = -2x_1 - x_2$

[18]

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1, x_2 \geq 0$

$x_1 + 2x_2 \leq 4$

using

Standard form and 2 phase method.

Min $z = a_1 + a_2 \Rightarrow$ Max $z = +a_1 + a_2 + ??$

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - s_2 + a_2 = 6$

$x_1 + 2x_2 + s_3 = 4$

$a_1, a_2, s_2, s_3 \geq 0$

a_1
 a_2
 s_3

Z	x_1	x_2	a_1	a_2	s_2	s_3	RHS:=t	b/a
1	0	0	1	1	0	0	0	
1	-7	-4	0	0	1	0	-9	
	<u>3</u>	1	1	0	0	0	3	1 →
	4	3	0	1	-1	0	6	3/2
	1	2	0	0	0	1	4	4
	↑							
1	0	-5/3	7/3	0	1	0	-2	
	1	1/3	1/3	0	0	0	1	3
	0	<u>5/3</u>	-4/3	1	-1	0	2	6/5 →
	0	5/3	-1/3	0	0	1	3	9/5
	↑							
1	0	0	1	1	0	0	0	
	1	0	3/5	-1/5	1/5	0	3/5	
	0	1	-4/5	3/5	-3/5	0	6/5	
	0	0	1	-1	1	1	1	

x_1
 a_2
 s_3

Phase I is completed \Rightarrow Proceed to phase II

Z	x_1	x_2	s_2	s_3	RHS	
1	2	1	0	0	0	
	1	0	$\frac{1}{5}$	0	315	
	0	1	$-\frac{3}{5}$	0	615	
	0	0	1	1	1	
1	0	0	$\frac{1}{5}$	0	-1215	- making basis 0
	1	0	$\frac{1}{5}$	0	315	
	0	1	$-\frac{3}{5}$	0	615	
	0	0	1	1	1	

The given solution is optimal

with $x_1 = 315$, $x_2 = 615$

$$\underline{\underline{Z_{\max} = -1215}}$$

≈ 14

SECTION - B

5. (a) (i) Form partial differential equation by eliminating function f from $z = y^2 + 2f(1/x + \log y)$
+ $\log y$.
(ii) Solve $(x-y)p + (x+y)q = 2xz$ [10]

$$i) z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \quad \text{--- (1)}$$

Diff wrt x and y .

$$\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

$$\therefore \frac{\partial z}{\partial y} = 2y + \frac{x}{y} \left(-\frac{x^2}{x} \frac{\partial z}{\partial x}\right)$$

$$\boxed{q = 2y - \frac{x^2}{y} p}$$

$$ii) (x-y)p + (x+y)q = 2xz$$

It is of Lagrange's Auxiliary equation form

$$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz}$$

$$\therefore \frac{dx}{dy} = \frac{x-y}{x+y}$$

Put $x = vy$

$$\text{If } \frac{dx}{dy} = \frac{(v-1)y}{(v+1)y}$$

$$\frac{dx}{dy} = \frac{v-1}{v+1}$$

$$(v+1)dv + 2dy = 0$$

$$-(v+1)^2 + 2y = 0$$

$$\boxed{u = \frac{(x^2 + y^2)^2}{2y^2} + 2y}$$

$$\frac{dx+dy}{2x} = \frac{dz}{2xz}$$

$$\therefore x+y = \ln z + \text{const}$$

$$\boxed{v = x+y - \ln z}$$

$\therefore f(u, v) = 0$ is
required solution

5. (b) Solve $(D^2 + DD' - 6D'^2)z = x^2 \sin(x+y)$.

[10]

$$(D^2 + DD' - 6D'^2)z = x^2 \sin(x+y)$$

Auxiliary equation is

$$(D^2 + DD' - 6D'^2)z = 0$$

$$\therefore (m^2 + m - 6)z = 0$$

$$\therefore (m+3)(m-2) = 0$$

$$\therefore m = -3, m = 2$$

$$\therefore z_c = \phi_1(y-3x) + \phi_2(y+2x)$$

putting $D = m, D' = 1$

Particular integral

$$(D+3D')(D-2D')z = x^2 \sin(x+y)$$

$$z_p = \frac{1}{(D+3D')(D-2D')} x^2 \sin(x+y)$$

put $y = c-x$

$$z_p = \frac{1}{D+3D'} \int x^2 \sin(c-x) dx$$

$$= \frac{1}{D+3D'} [x^2 \cos x]$$

5. (c) A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's $\frac{1}{3}$ rd rule, find the velocity of the rocket at t = 80 seconds.

t(sec):	0	10	20	30	40	50	60	70	80
f(cm/sec ²):	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

$$v = \int_0^t f(t) dt$$

[10]

Simpson's $\frac{1}{3}$ rule

$$v = \frac{h}{3} \left[y_0 + y_n + 2(y_2 + y_4 + y_6 + \dots) + 4(y_1 + y_3 + \dots) \right]$$

$$v = \frac{10}{3} \left[30 + 50.67 + 4(31.63 + 35.47 + 40.33 + 46.69) + 2(33.34 + 37.75 + 43.25) \right]$$

$$v = 3090.1 \text{ cm/sec}$$

3090.1 cm/sec

5. (d) (i) Simplify the expression $A = XY + \overline{XZ} + X\overline{Y}Z(XY + Z)$

(ii) Simplify the Boolean expression $Y = \overline{A \cdot B} + \overline{A} + B$

Prepare truth table to show that the simplified expression is correct. [10]

$$Y = XY + \overline{XZ} + X\overline{Y}Z(XY + Z)$$

$$Y = XY + \overline{XZ} + X\overline{Y}Z + X\overline{Y}Z$$

→ Distributive law

$$XX = X$$

$$Y = XY + \overline{XZ} + X\overline{Y}Z$$

$$\rightarrow X + X = X$$

$$Y = X(\overline{Y} + \overline{Y}Z) + \overline{XZ}$$

→ Distributive law

$$Y = X[(\overline{Y} + \overline{Y})Z] + \overline{XZ}$$

→ Distributive law

$$Y = X[\overline{Y} + Z] + \overline{XZ}$$

$$Y = XY + XZ + \overline{XZ}$$

$$Y = XY + 1 = 1$$

$$\text{ii) } Y = \overline{A \cdot B} + \overline{A + B}$$

$$Y = \overline{A} + \overline{B} + A \cdot B \quad \text{--- De Morgan's law.}$$

$$Y = \overline{A} + A \cdot \overline{B} + \overline{B}$$

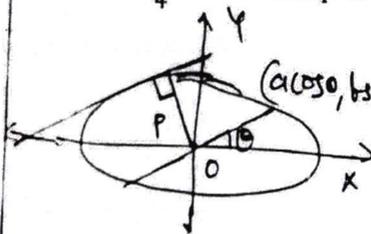
$$Y = (\overline{A} + A) \cdot (\overline{A} + \overline{B}) + \overline{B} =$$

$$Y = \overline{A} + \overline{B} + \overline{B} = \overline{A} + \overline{B}$$

A	B	AB	\overline{A}	\overline{B}	$\overline{A+B}$	$\overline{\overline{A+B}}$	$\overline{AB} + \overline{A+B}$	$\overline{A+B}$
0	0	0	1	1	1	0	1	1
0	1	0	1	0	1	0	1	1
1	0	0	0	1	0	1	1	1
1	1	1	0	0	0	0	0	0

Hence proved

5. (c) Show that the M.I. of an ellipse of mass M and semi-axes a and b about a tangent is $\frac{5}{4}Mp^2$, where p is the perpendicular from the centre on the tangent. [10]



$(a \cos \theta, b \sin \theta)$ Tangent to ellipse is given by

$$y = mx + \sqrt{m^2 a^2 + b^2}$$

Now M.I. of ellipse @ OX axis is

$$A = \frac{Mb^2}{4}$$

$$\text{MI @ OY axis} = \frac{Ma^2}{4} = B$$

$$PI = 0 = F$$

\therefore symmetric

MI @ line through O and parallel to tangent of slope m ($m = \tan \theta$)

$$I = A \cos^2 \theta + B \sin^2 \theta - 2F \cos \theta \sin \theta$$

$$I = A \cos^2 \theta + B \sin^2 \theta =$$

$$I = \frac{M}{4} (b^2 \cos^2 \theta + a^2 \sin^2 \theta) \quad \text{--- (1)}$$

Now perpendicular dist p of tangent from $O(0,0)$

$$p = \frac{\left| \frac{\sqrt{m^2 a^2 + b^2}}{\sqrt{m^2 + 1}} \right|}{\left| \frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{\sin^2 \theta + \cos^2 \theta} \right|}$$

$$\therefore p = \frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{\sin^2 \theta + \cos^2 \theta} \quad \text{--- (2)}$$

$$\therefore MI @ \text{ tangent} = MI @ \text{ parallel axis through centre} + MI @ MP$$

$$= \frac{Mp^2}{4} + Mp^2 \quad \text{--- FROM (1) \& (2)}$$

$$= \frac{5Mp^2}{4}$$

6. (a) (i) Find a complete integral of $z^2 = pqxy$.

(ii) Reduce the equation

$\partial^2 z / \partial x^2 + 2(\partial^2 z / \partial x \partial y) + \partial^2 z / \partial y^2 = 0$ to canonical form and hence solve it.

[6+12=18]

$$z^2 = pqxy$$

$$\therefore 1 = \left(\frac{px}{z} \right) \left(\frac{qy}{z} \right) \quad \text{--- (1)}$$

$$\frac{1}{z} dz = dx$$

$$\frac{1}{z} dz = dz$$

$$\frac{1}{y} dy = dy$$

$$\ln z = x$$

$$\ln z = z$$

$$\ln y = y$$

$$\therefore (1) = 1 = p \phi$$

$$\text{Put } p = a \quad \therefore \phi = \frac{1}{a}$$

$$\therefore dz = a dx + \frac{1}{a} dy$$

$$z = ax + \frac{1}{a} y \Rightarrow$$

$$\ln z = a \ln x + \frac{1}{a} \ln y$$

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{--- (1)}$$

$$r + 2s + t = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)^2 = 0 \quad \lambda = -1, -1$$

$$\frac{dy}{dx} - 1 = 0 \Rightarrow dy = dx$$

$$y = x + u$$

$$u = y - x$$

$$\text{let } v = x$$

$$\therefore \begin{vmatrix} y-x & x \\ 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \text{ hence } v \text{ is allowed}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} (-1) + \frac{\partial z}{\partial v} (1) \quad \left| \frac{\partial}{\partial x} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right.$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} (1) \quad \left| \frac{\partial}{\partial y} = \frac{\partial}{\partial u} \right.$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left[-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right] = \left[-\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right] \left[-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right]$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{--- (2)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial z}{\partial u} \right] = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}$$

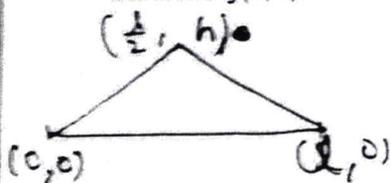
$$\left(\frac{\partial^2 z}{\partial x \partial y} = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} \right) \quad \text{--- (3)}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} \quad \text{--- (4)}$$

$$\frac{\partial^2 z}{\partial v^2} = 0 \Rightarrow$$

97. Please complete it by integrating.

7. (a) A taut string of length l has its ends $x = 0$ and $x = l$ fixed. The midpoint is taken to a small height h and released from rest at time $t = 0$. Find the displacement function $y(x, t)$. (20)



Let $y(x, t)$ be displacement

Boundary condition

$$y(0, t) = 0, \quad y(l, t) = 0 \quad \text{--- (1)}$$

Initial condition

$$\frac{\partial y}{\partial t}(x, 0) = 0$$

$$y(x, 0) = \frac{2h}{l}x \quad 0 \leq x \leq \frac{l}{2}$$

$$= \frac{2h(l-x)}{l} \quad \frac{l}{2} \leq x \leq l$$

--- (2)

Equation of string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$\text{Let } y(x, t) = X(x)T(t)$$

$$\therefore XT'' = c^2 X''T$$

$$\therefore \frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda^2$$

$$\frac{X''}{X} = -\lambda^2$$

$$(D^2 + \lambda^2)X = 0$$

$$\therefore D^2 = -\lambda^2$$

$$D = \pm \lambda i$$

$$X = A \cos \lambda x + B \sin \lambda x$$

$$X(0) = 0, \quad X(l) = 0 \Rightarrow \text{From (1)}$$

$$0 = A + B$$

$$0 = B \sin \lambda l$$

$$\therefore \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}$$

$$\frac{T''}{c^2 T} = -\lambda^2$$

$$T'' = -c^2 \lambda^2 T$$

$$D^2 = \pm c \lambda i$$

$$T = C \cos c \lambda t + D \sin c \lambda t$$

$$T = \sum_{n=1}^{\infty} [C_n \cos\left(\frac{n\pi c}{l}t\right) + D_n \sin\left(\frac{n\pi c}{l}t\right)]$$

$$x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

$$\therefore y = \sum \left[E_n \cos \left(\frac{n\pi ct}{l} \right) + F_n \sin \left(\frac{n\pi ct}{l} \right) \right] \sin \frac{n\pi x}{l}$$

subjecting to (2).

$$y'(x, t) = \left[E_n \sin \left(\frac{n\pi ct}{l} \right) + F_n \cos \left(\frac{n\pi ct}{l} \right) \right] \left(\frac{n\pi c}{l} \right) \left(\sin \frac{n\pi x}{l} \right)$$

$$y'(x, 0) = 0 = F_n \left(\frac{n\pi c}{l} \right) \sin \left(\frac{n\pi x}{l} \right) \Rightarrow F_n = 0$$

$$\therefore y(x, 0) = \sum E_n \cos \left(\frac{n\pi ct}{l} \right) \sin \left(\frac{n\pi x}{l} \right) = \sum E_n \sin \frac{n\pi x}{l}$$

By Fourier series

$$E_n = \frac{2}{l} \int_0^l y(x) \sin \left(\frac{n\pi x}{l} \right) dx =$$

$$= \frac{2}{l} \left[\int_0^{l/2} \frac{2hx}{l} \sin \left(\frac{n\pi x}{l} \right) dx + \int_{l/2}^l \frac{2h(l-x)}{l} \sin \left(\frac{n\pi x}{l} \right) dx \right]$$

$$\int_0^{l/2} \frac{2hx}{l} \sin \left(\frac{n\pi x}{l} \right) dx = \frac{2h}{l} \left[\left(\frac{-x \cos \left(\frac{n\pi x}{l} \right)}{\frac{n\pi}{l}} \right) + \left[\frac{-1 \sin \left(\frac{n\pi x}{l} \right)}{\left(\frac{n\pi}{l} \right)^2} \right] \right]_0^{l/2}$$

$$= \frac{2h}{l} \left[\frac{-l^2}{2n\pi} \cos \left(\frac{n\pi}{2} \right) + \frac{\sin \left(\frac{n\pi}{2} \right)}{n^2 \pi^2} \right]$$

$$\int_{l/2}^l \frac{2h(l-x)}{l} \sin \left(\frac{n\pi x}{l} \right) dx = \frac{2h}{l} \left\{ \left[\frac{-(l-x) \cos \left(\frac{n\pi x}{l} \right)}{\frac{n\pi}{l}} \right] - \left[\frac{\sin \left(\frac{n\pi x}{l} \right)}{\left(\frac{n\pi}{l} \right)^2} \right] \right\}_{l/2}^l$$

$$= \frac{2h}{l} \left[\frac{l^2}{2n\pi} \cos \left(\frac{n\pi}{2} \right) + \frac{\sin \left(\frac{n\pi}{2} \right) l^2}{n^2 \pi^2} \right]$$

$$E_n = \frac{16h \sin \left(\frac{n\pi}{2} \right)}{n^2 \pi^2} = (-1)^{m+1} \frac{16h}{\pi^2 (2m+1)^2}$$

$$\therefore y(x, t) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{16h}{\pi^2 (2m+1)^2} \cos \left[\frac{(2m+1)\pi ct}{l} \right] \sin \frac{(2m+1)\pi x}{l}$$

7. (b) (i) Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at

$x = 0.2, 0.4.$

(ii) Convert 1011101.1011 to octal and then to hexadecimal. [12+5=17]

$$\frac{dy}{dx} = f(x, y) = \frac{y^2 - x^2}{y^2 + x^2} \quad h = 0.2$$

$$k_1 = h [f(x_0, y_0)] = h [f(0, 1)] = 0.2$$

$$k_2 = h [f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})] = h f[0.1, 1.1] = 0.1967$$

$$k_3 = h [f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})] = h f[0.1, 1.09836] = 0.1967$$

$$k_4 = h [f(x_0 + h, y_0 + k_3)] = h f[0.2, 1.1967] = 0.18913$$

$$k = \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4] = 0.19598$$

$$y_1 = y_0 + k$$

$$y(0.2) = 1.1960$$

Step 2

$$k_1 = h f[0.2, 1.1960] = 0.18913$$

$$k_2 = h f[0.3, 1.1960 + \frac{k_1}{2}] = 0.17951$$

$$k_3 = h f[0.3, 1.1960 + \frac{k_2}{2}] = 0.17934$$

$$k_4 = h f[0.4, 1.1960 + k_3] = 0.1688$$

$$\therefore k = \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4] = 0.17927$$

$$y(0.4) = 1.3752$$

$$\text{ii) } \frac{101101}{1} \cdot \frac{101100}{35} = \underline{\underline{(135.54)_8}}$$

$$(135.54) = 1 \times 8^2 + 3 \times 8 + 5 \times 1 + 5 \times \frac{1}{8} + 4 \times \frac{1}{8^2}$$

$$= 9?$$

$$\frac{10101}{5} \cdot \frac{1101}{13} \cdot \frac{1011}{11} = \underline{\underline{(5D.B)_{16}}}$$

7. (c) Prove that the velocity potentials $\phi_1 = x^2 - y^2$ and $\phi_2 = r^{1/2} \cos(\theta/2)$ are solutions of the Laplace equation and the velocity potential $\phi_3 = (x^2 - y^2) + r^{1/2} \cos(\theta/2)$ satisfies $\nabla^2 \phi_3 = 0$. [13]

$$\phi_1 = x^2 - y^2$$

$$\frac{\partial \phi_1}{\partial x} = 2x \quad \frac{\partial \phi_1}{\partial y} = -2y$$

$$\frac{\partial^2 \phi_1}{\partial x^2} = 2 \quad \frac{\partial^2 \phi_1}{\partial y^2} = -2$$

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0$$

$$\phi_2 = \sqrt{r} \cos(\theta/2)$$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$$

$$\frac{\partial^2 \phi_2}{\partial r^2} = \frac{1}{2\sqrt{r}} \cos\left(\frac{\theta}{2}\right)$$

$$\frac{\partial^2 \phi}{\partial r^2} = \frac{-1}{4\sqrt{r}} \cos\frac{\theta}{2}$$

$$\frac{\partial \phi}{\partial \theta} = -\sqrt{r} \sin\left(\frac{\theta}{2}\right) \left(\frac{1}{2}\right)$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = \frac{-1}{4} \sqrt{r} \cos\left(\frac{\theta}{2}\right)$$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

→ satisfies Laplace equation

$$\phi_3 = \phi_1 + \phi_2$$

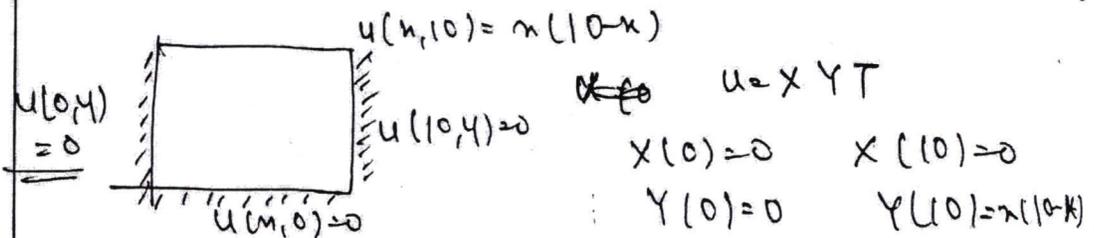
$$\therefore \nabla^2 \phi_3 = \nabla^2 \phi_1 + \nabla^2 \phi_2$$

$$\nabla^2 \phi_3 = 0 + 0$$

$$\boxed{\nabla^2 \phi_3 = 0}$$

$\therefore \phi_1$ and ϕ_2 satisfy Laplace equation

8. (a) A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 10$ and $y = 10$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 10) = x(10 - x)$ while the other three faces are kept at 0°C . Find the steady state temperature in the plane. [17]



2D Heat wave equation is

$$\frac{\partial u}{\partial t} = m^2 \frac{\partial^2 u}{\partial x^2} + n^2 \frac{\partial^2 u}{\partial y^2}$$

\therefore At steady state $\frac{\partial u}{\partial t} = 0$

$$\therefore m^2 \frac{\partial^2 u}{\partial x^2} + n^2 \frac{\partial^2 u}{\partial y^2} = 0$$

$$u = X(x) Y(y)$$

$$\therefore m^2 X'' Y + n^2 X Y'' = 0$$

$$\therefore \frac{m^2 X''}{X} + \frac{n^2 Y''}{Y} = 0 \Rightarrow \frac{m^2 X''}{X} = -\frac{n^2 Y''}{Y} = K$$

$$\frac{d^2 X}{dx^2} = \frac{K}{m^2}$$

$$\frac{d^2 Y}{dy^2} = \frac{-K}{n^2}$$

$$D^2 = \frac{K}{m^2} X$$

$$\frac{d^2 Y}{dy^2} = \frac{D^2}{n^2}$$

$$K < 0$$

otherwise $u = 0$

$$@ x \neq 0 \quad D = 0,$$

$$K = -\lambda^2$$

$$D^2 = \frac{-\lambda^2}{m^2} X$$

$$D^2 = \frac{\lambda^2}{n^2}$$

$$D^1 = \pm \frac{\lambda}{n}$$

$$\therefore Y = C_1 e^{\frac{\lambda}{n} y} + D e^{-\frac{\lambda}{n} y}$$

$$D = \pm \frac{\lambda}{m}$$

$$Y(0) = 0 \Rightarrow$$

$$\therefore C + D = 0$$

$$\therefore C = -D$$

$$X = A \cos \frac{\lambda}{m} x + B \sin \frac{\lambda}{m} x$$

$$X(0) = 0 \Rightarrow A = 0$$

$$X(10) = 0 \Rightarrow B \sin \frac{\lambda \cdot 10}{m} = 0$$

$$\frac{\lambda \cdot 10}{m} = n\pi$$

$$Y(10) = x(10-x)$$

$$\Rightarrow D e^{\frac{\lambda}{n}(10)} + D e^{-\frac{\lambda}{n}(10)}$$

$$= x(10-x)$$

$$X(x) = \sum B_n \sin\left(\frac{n\pi x}{10}\right)$$

$$\therefore D = \frac{x(10-x)}{e^{\frac{\lambda}{n}(10)} - e^{-\frac{\lambda}{n}(10)}}$$

$$Y = \frac{x(10-x)}{(e^{\frac{\lambda \cdot 10}{n}} - e^{-\frac{\lambda \cdot 10}{n}})} \left(-e^{\frac{\lambda y}{n}} + e^{-\frac{\lambda y}{n}} \right)$$

$$u = XY$$

$$\therefore u = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \frac{n(10-w) \left(e^{-\frac{ny}{L}} - e^{\frac{ny}{L}} \right)}{\left(e^{-\frac{10y}{L}} - e^{\frac{10y}{L}} \right)}$$

Please refer key

is required condition at steady state

8. (b) (i) Convert hexadecimal number 2647 to octal.
 (ii) Convert hexadecimal number 4A.67 to binary.
 (iii) A committee of three approves proposal by majority vote. Each member can vote for the proposal by pressing a button at the side of their chairs. These three buttons are connected to a light bulb. For a proposal whenever the majority of votes takes place, a light bulb is turned on. Design a circuit as simple as possible so that the current passes and the light bulb is turned on only when the proposal is approved. [3+3+10=16]

$$(\text{26}) \quad (2647)_{16} = \cancel{01001000111}$$

$$\left(\begin{array}{cccc} \underline{0010} & \underline{0110} & \underline{0100} & \underline{0111} \\ 2 & 6 & 4 & 7 \end{array} \right)_L$$

$$\cancel{00010011001000111}$$

$$(2647)_{16} = \underline{\underline{(23107)_8}}$$

ii) $(4A \cdot 67)_{16} =$

$$4A \cdot 67 = \frac{0100}{4} \frac{1010}{A} \cdot \frac{0110}{6} \frac{0111}{7}$$

$(4A \cdot 67)_{16} = (1001010.01100111)$

iii) Minimum 2 votes are required for proposal

~~$Y = AB + BC$~~

Let A, B, C denote approvals by ~~an~~ persons individually

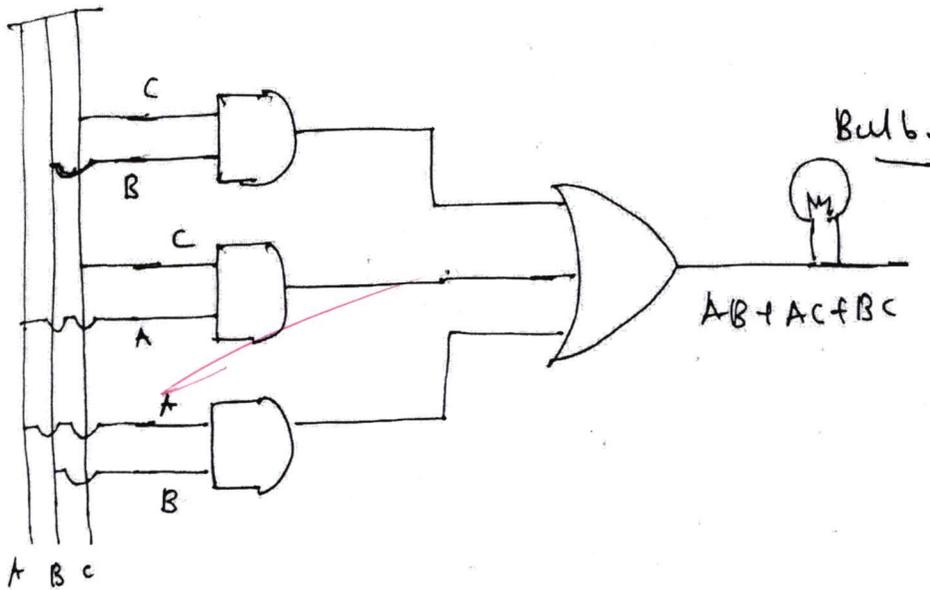
$ABC + A\bar{B}C + \bar{A}BC + ABC$

$Y = AB + AC + BC + ABC$

$= AB(C+1) + AC + BC$

$= AB + AC + BC$

By distributive law



8. (c) A sphere of radius R , whose centre is at rest, vibrates radially in an infinite incompressible fluid of density ρ , which is at rest at infinity. If the pressure at infinity is Π , show that the pressure at the surface of the sphere at time t is

$$\Pi + \frac{1}{2} \rho \left\{ \frac{d^2 R^2}{dt^2} + \left(\frac{dR}{dt} \right)^2 \right\}. \quad [17]$$

The equation of motion is given by

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} = F - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad \text{--- (1)}$$

Equation of continuity for sphere is $x^2 v = f(t)$ --- (2)

$$\therefore v = \frac{f(t)}{x^2} \Rightarrow \frac{\partial v}{\partial t} = \frac{f'(t)}{x^2}$$

$$\therefore \frac{f'(t)}{x^2} + \frac{1}{2} \frac{\partial v^2}{\partial x} = 0 - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

Integrating w.r.t x on both sides

$$\frac{f'(t)}{-x} + \frac{1}{2} v^2 = -\frac{p}{\rho} + C$$

$$\text{@ } x = \infty, v = 0, p = \Pi \Rightarrow C = \frac{\Pi}{\rho}$$

$$\therefore \frac{f'(t)}{-x} + \frac{1}{2} v^2 = \frac{\Pi - p}{\rho}$$

$$\text{@ } x = R, v = \dot{R}, p = P$$

$$\therefore R^2 \dot{R} = f(t) \Rightarrow$$

$$\therefore f'(t) = 2R \dot{R}^2 + R^2 \ddot{R}$$

$$\therefore \frac{2R \dot{R}^2 + R^2 \ddot{R}}{-R} + \frac{1}{2} \dot{R}^2 = \frac{\Pi - P}{\rho}$$

$$\therefore -\frac{3}{2} \dot{R}^2 - R \ddot{R} = \frac{\Pi - P}{\rho}$$

$$P = \pi + \rho \left(\frac{3}{2} \dot{R}^2 + R \ddot{R} \right) \quad \text{--- (3)}$$

$$\begin{aligned} \text{Now } \frac{d^2 R^2}{dt^2} &= \frac{d}{dt} \left(\frac{dR^2}{dt} \right) = \frac{d}{dt} [2R \dot{R}] \\ &= 2R \ddot{R} + 2\dot{R}^2 \end{aligned}$$

$$\frac{d^2 R^2}{dt^2} + \left(\frac{dR}{dt} \right)^2 = 2R \ddot{R} + 3\dot{R}^2 \quad \text{--- (4)}$$

$$\therefore P = \pi + \frac{1}{2} \rho (3\dot{R}^2 + 2R \ddot{R})$$

$$P = \pi + \frac{1}{2} \rho \left[\frac{d^2 R^2}{dt^2} + \left(\frac{dR}{dt} \right)^2 \right] \quad \text{--- from (4)}$$

Hence proved

ROUGH SPACE

$$\frac{\partial \phi}{\partial r} = \sqrt{r} \cos \frac{\theta}{2}$$

$$\frac{\partial \phi}{\partial r} = \frac{1}{2\sqrt{r}} \cos \frac{\theta}{2}$$

$$\frac{\partial \phi}{\partial r^2} = \frac{1}{2} \times \frac{1}{r} \times \frac{1}{r^{3/2}} \cos \frac{\theta}{2}$$

$$\frac{\partial \phi}{\partial r^2} = \frac{1}{4r^{3/2}} \cos \frac{\theta}{2}$$

$$\frac{1}{r} \frac{\partial \phi}{\partial r} =$$

$$\frac{\partial \phi}{\partial \theta} = -\sqrt{r} \sin \frac{\theta}{2} \times \frac{1}{2}$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = -\frac{\sqrt{r}}{4} \times \cos \frac{\theta}{2}$$

$$\frac{1}{r^2} \frac{\partial \phi}{\partial \theta^2} = -\frac{\sqrt{r}}{4r^{3/2}} \cos \frac{\theta}{2}$$

$$-\frac{1}{8r^{3/2}} \cos \frac{\theta}{2}$$