SECTION 'A'

Q. 1. **Answer any *five* of the following :**

(a) Let S be the set of all real numbers except -1. Define $*$ on S by

$$a * b = a + b + ab$$

Is $(S, *)$ a group ?

Find the solution of the equation

$$2 * x * 3 = 7 \text{ in } S.$$
12

(b) If G is a group of real numbers under addition and N is the subgroup of G consisting of integers, prove that G/N is isomorphic to the group H of all complex numbers of absolute value 1 under multiplication.
12

(c) Examine the convergence of

$$\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$$
12

(d) Prove that the function f defined by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$$

is nowhere continuous.
12

(e) Determine all bilinear transformations which map the half plane Im $(z) \geq 0$ into the unit circle $|w| \leq 1$.
12

(f) Given the programme

$$\text{Maximize } u = 5x + 2y$$
$$\text{subject to } x + 3y \leq 12$$
$$3x - 4y \leq 9$$
$$7x + 8y \leq 20$$

1

$$x, y \geq 0$$

Write its dual in the standard form. **12**

Q. 2. (a) (i) Let O (G) = 108. Show that there exists a normal subgroup or order 27 or 9.

(ii) Let G be the set of all those ordered pairs (a, b) of real numbers for which a ≠ 0 and define in G, an operation $\otimes$ as follows :

$$(a, b) \otimes (c, d) = (ac, bc + d)$$

Examine whether G is a group w.r.t. the operation $\otimes$. If it is a group, is G abelian ? **10**

(b) Show that

$$Z[\sqrt{2}] = \{a + \sqrt{2}\, b \mid a, b \in Z\}$$

is a Euclidean domain. **30**

Q. 3. (a) A twice differentiable function f is such that f (a) = f (b) = 0 and f (c) > 0 for a < c < b. Prove that there is at least one value ξ, a < ξ < b for which f″ (ξ) < 0. **20**

(b) Show that the function given by

$$f(x,y) = \begin{cases} \dfrac{x^3 + 2y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & , & (x, y) > (0, 0) \end{cases}$$

(i) is continuous at (0, 0).

(ii) possesses partial derivatives

f_x (0, 0) and f_y (0, 0). **20**

(c) Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Q. 4. (a) With the aid of residues, evaluate

$$\int_0^\pi \frac{\cos 2\theta}{1 - 2a \cos\theta + a^2}\, d\theta , \; -1 < a < 1$$

15

2

(b) Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.   15

(c) Use the simplex method to solve the problem

         Maximize $u = 2x + 3y$

         subject to $-2x + 3y \leq 2$

                $3x + 2y \leq 5$

                $x, y \geq 0$ 30

SECTION 'B'

Q. 5.   **Answer any *five* of the following :**

(a) Solve :

$$px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$$ 12

(b) Solve :

$$\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 2 \sin(3x + 2y)$$ 12

(c) Evaluate

$$I = \int_0^1 e^{-x^2} dx$$

by the Simpson's rule

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + 2f(x_2)]$$

$$+ 4f(x_3) + \ldots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]$$

with

$2n = 10, \Delta x = 0.1, x_0 = 0, x_1 = 0.1, \ldots, x_{10} = 1.0$ 12

(d)   (i) Given the number 59.625 in decimal system. Write its binary equivalent.   6

         (ii) Given the number 3898 in decimal system. Write its equivalent in system base 8.   6

(e) Given points A (0, 0) and B (x_0, y_0) not in the same vertical,

it is required to find a curve in the $x - y$ plane joining A to B so that a particle starting from rest will traverse from A to B along this curve without friction in the shortest possible time. If $y = y(x)$ is the required curve find the function $f(x, y, z)$ such that the equation of motion can be written as

$$\frac{dx}{dt} = f(x, y(x), y'(x)).$$
12

(f) A steady inviscid incompressible flow has a velocity field
$$u = fx, \quad v = -fy, \quad w = 0$$
where f is a constant. Derive an expression for the pressure field p $\{x, y, z\}$ if the pressure

$$p\{0, 0, 0\} = p_0 \text{ and } \vec{g} = -g\,\vec{i}_z.$$
12

Q. 6. (a) The deflection of a vibrating string of length l, is governed by the partial differential equation $u_{tt} = C^2 u_{xx}$. The ends of the string are fixed at $x = 0$ and l. The initial velocity is zero. The initial displacement is given by

$$u(x, 0) = \frac{x}{l}, \quad 0 < x < \frac{l}{2}$$

$$= \frac{1}{l}(l - x), \quad \frac{l}{2} < x < l.$$

Find the deflection of the string at any instant of time. 30

(b) Find the surface passing through the parabolas $z = 0$, $y^2 = 4ax$ and $z = 1$, $y^2 = -4ax$ and satisfying the equation

$$x\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial z}{\partial x} = 0$$
15

(c) Solve the equation

$$p^2 x + q^2 y = z, \quad p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

by Charpit's method. **15**

Q. 7. (a) If Q is a polynomial with simple roots $\alpha_1, \alpha_2, ... \alpha n$ and if P is a polynomial of degree $< n$, show that

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(x - \alpha_k)}.$$

Hence prove that there exists a unique polynomial of degree $< n$ with given values c_k at the point α_k, $k = 1, 2, ... n$. **30**

(b) Draw a programme outline and a flow chart and also write a programme in BASIC to enable solving the following system of 3 linear equations in 3 unknowns x_1, x_2 and x_3:

$$C * X = D$$

with

$$C = (c_{ij})^3_{i, j = 1}, X = (x_j)^3_{j = 1}, D = (d_i)^3_{i = 1}. \qquad \textbf{30}$$

Q. 8. (a) A particle of mass m is constrained to move on the surface of a cylinder. The particle is subjected to a force directed towards the origin and proportional to the distance of the particle from the origin. Construct the Hamiltonian and Hamilton's equations of motion. **30**

(b) Liquid is contained between two parallel planes, the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated; prove that if P be the pressure at the outer surface, the initial pressure at any point on the liquid, distant r from the centre is

$$P \frac{\log r - \log b}{\log a - \log b} \qquad \textbf{30}$$