C.S.E. (MAIN)

MATHEMATICS—2004

(PAPER–II)

Time allowed : 3 hours *Max. Marks : 300*

INSTRUCTIONS

Each question is printed both in Hindi and in English.

Answers must be written in the medium specified in the Admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Questions 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each Section.

Assume suitable data if considered necessary and indicate the same clearly.

All questions carry equal marks.

SECTION 'A'

Q. 1. Answer any *five* of the following :

(a) If p is a prime number of the form $4n + 1$, n being a natural number, then show that congruence $x^2 \equiv -1 \bmod p$ is solvable. 12

(b) Let G be a group such that of all $a, b, \in G$

(i) $ab = ba$ (ii) $(O(a), O(b)) = 1$ then show that $O(ab) = O(a)\,O(b)$. 12

(c) Show that the function $f(x)$ defined as :

$$f(x) = \frac{1}{2^n}, \frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}, n = 0, 1, 2, \ldots.$$

$$f(0) = 0$$

is integrable in [0, 1], although it has an infinite number of points of discontinuity. Show that

$$\int_0^1 f(x)dx = \frac{2}{3}.$$ 12

(d) Show that the function $f(x)$ defined on R by :

$$f(x) = \begin{cases} x \text{ when } x \text{ is irrational} \\ -x \text{ when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$. 12

(e) Find the image of the line $y = x$ under the mapping $w = \dfrac{4}{z^2+1}$ and draw the same. Find the points where this transformation ceases to be conformal. 12

(f) Use Simplex method to solve the linear programming problem:

Max. $z = 3x_1 + 2x_2$,

subject to $x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$. 12

Q. 2. (a) Verify that the set E of the four roots of $x^4 - 1 = 0$ forms a multiplicative group. Also prove that a transformation T, T $(n) = i^n$

is a homomorphism from I_+ (Group of all integers with addition) onto E under multiplication. 10

(b) Prove that if the cancellation law holds for a ring R then $a\ (\neq 0) \in$ R is not a zero divisor and conversely.

(c) The residue class ring $\dfrac{Z}{(m)}$ is a field iff m is a prime integer.

15

(d) Define irreducible element and prime element in an integral domain D with units. Prove that every prime element in D is irreducible and converse of this is not (in general) true. 25

Q. 3. (a) If (x, y, z) be the lengths of perpendiculars drawn from any interior point P of a triangle ABC on the sides BC, CA and AB respectively, then find the minimum value of $x^2 + y^2 + z^2$, the sides of the triangle ABC being a, b, c. 20

(b) Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ax$ and the plane $z = 0$. 20

(c) Let $f(x) \geq g(x)$ for every x in $[a, b]$ and f and g are both bounded and Riemann integrable on $[a, b]$. At a point $c \in [a, b]$, let f and g be continuous and $f(c) > g(c)$ then prove that

$$\int_a^b f(x)dx > \int_a^b g(x)dx$$ and hence show that

$$-\frac{1}{2} < \int_a^b \frac{x^3 \cos 5x}{2+x^2}\, dx < \frac{1}{2}$$ 20

Q. 4. (a) If all zeroes of a polynomial P (z) lie in a half plane then show that zeroes of the derivative $P'(z)$ also lie in the same half plane. 15

(b) Using contour integration evaluate

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta + p^2}\, d\theta, \quad 0 < p < 1$$ 15

(c) A travelling salesman has to visit 5 cities. He wishes to start from a particular city, visit each city once and them return to his starting point. Cost of going from one city to another is given below :

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 4 & 10 & 14 & 2 \\ 12 & \infty & 6 & 10 & 4 \\ 16 & 14 & \infty & 8 & 14 \\ 24 & 8 & 12 & \infty & 10 \\ 2 & 6 & 4 & 16 & \infty \end{bmatrix} \end{array}$$

You are required to find the least cost route.

(d) A department has 4 technicians and 4 tasks are to be performed. The technicians differ in efficiency and tasks differ in their intrinsic difficulty. The estimate of time (in hours), each technician would take to perform a task is given below. How should the tasks be allotted, one to a technician, so as to minimize the total work hours ? 15

Technician \ Task	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

SECTION 'B'

Q. 5. Attempt any *five* of the following :

(a) Find the integral surface of the following partial differential equation :

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z \qquad 12$$

(b) Find the complete integral of the partial differential equation $(p^2+q^2)x = pz$ and deduce the solution which passes through the curve $x=0,\ z^2=4y$. 12

(c) The velocity of a particle at distance S from a point on its path is given by the following table :

S (meters)	V (m / sec)
0	47
10	58
20	64
30	65
40	61
50	52
60	38

Estimate the time taken to travel the first 60 meters using Simpson's $\frac{1}{3}$ rule. Compare the result with Simpson's $\frac{3}{8}$ rule. 12

(d) (i) If $(AB, CD)_{16} = (x)_2 = (y)_8 = (z)_{10}$ then find x, y and z. 6

(ii) In a 4-bit representation, what is the value of 1111 in signed integer form, unsigned integer form, signed 1's complement form and signed 2's complement form ? 6

(e) A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$ which is assumed frictionless. Obtain the equation of motion. Show that it will describe a horizontal circle in the plane $z = h$, provided that it is given an angular velocity whose magnitude is $\omega = \sqrt{2g / a}$. 12

(f) In an incompressible fluid, the vorticity at every point is constant, in magnitude and direction. Do the velocity components satisfy the Laplace equation ? Justify. 12

Q. 6. (a) Solve the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (y-1)e^x. \qquad 15$$

(b) A uniform string of length l, held tightly between $x = 0$ and $x = l$ with no initial displacement, is struck at $x = a$, $0 < a < l$, with velocity v_0. Find the displacement of the string at any time $t > 0$. 30

(c) Using Charpit's method, find the complete solution of the partial differential equation $p^2x + q^2y = z$. 15

Q. 7. (a) How many positive and negative roots of the equation $e^x - 5 \sin x = 0$ exist ? Find the smallest positive root correct to 3 decimals, using Newton-Raphson method. 15

(b) Using Gauss-Siedel iterative method, find the solution of the following system :

$4x - y + 8z = 26$

$5x + 2y - z = 6$

$x - 10y + 2z = -13$

upto three iterations. 15

(c) In a certain exam, candidates have to take 2 papers under part A and 2 papers under part B. A candidate has to obtain minimum of 40% in each paper under part A, with an average of 50%, together with a minimum of 35% in each paper under part B, with an average of 40%. For a complete PASS, an overall minimum of 50% is required. Write a BASIC program to declare the result of 100 candidates. 15

(d) Write a BASIC program for solving the differential equation

$$\frac{dy}{dx} = x^2 + y^2, y(0) = 0.1$$

to get $y(x)$, for $0.2 \leq x \leq 5$ at an equal interval of 0.2, by Runge-Kutta fourth order method. 15

Q. 8. (a) Derive the Hamilton equations of motion from the principle of least action and obtain the same for a particle of mass m moving in a force field of potential V.

Write these equations in spherical coordinates (r, θ, ϕ). 30

(b) The space between two infinitely long coaxial cylinders of radii a and b $(b > a)$ respectively is filled by a homogeneous fluid, of density ρ. The inner cylinder is suddenly moved with velocity v perpendicular to this axis, the outer being kept fixed. Show that the resulting impulsive pressure on a length l of inner cylinder is,

$$\pi\rho a^2 l \frac{b^2 + a^2}{b^2 - a^2} v.$$ 30

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