C.S.E. (MAIN)
MATHEMATICS—2004
(PAPER–I)

Time allowed : 3 hours *Max. Marks : 300*

INSTRUCTIONS

Each question is printed both in Hindi and in English.

Answers must be written in the medium specified in the Admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Questions 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each Section.

Assume suitable data if considered necessary and indicate the same clearly.

All questions carry equal marks.

SECTION 'A'

Q. 1. Attempt any *five* of the following :

(a) Let S be space generated by the vectors $\{(0, 2, 6), (3, 1, 6), (4, -2, -2)\}$. What is the dimension of the space S ? Find a basis for S. 12

(b) Show that $f : \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation, where $f(x, y, z) = 3x + y - z$. What is the dimension of the kernel ? Find a basis for the kernel. 12

(c) Prove that the function f defined on $[0, 4]$ by $f(x) = [x]$, greatest integer $\le x$, $x \in [0, 4]$ is integrable on $[0, 4]$ and that $\int_0^4 f(x)dx = 6.$

12

(d) Shaw that : $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}, x > 0.$ 12

(e) Prove that the locus of the foot of the perpendicular drawn from the vertex on a tangent to the parabola $y^2 = 4ax$ is $(x+a)y^2 + x^3 = 0.$

12

(f) Find the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$, which are parallel to the plane $2x + y - z = 4.$

12

Q. 2. (a) Show that the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$

which is represented by the matrix $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$ is one-to-one.

Find a basis for its image. 15

(b) Verify whether the following system of equations is consistent :

$$x + 3z = 5$$
$$-2x + 5y - z = 0$$
$$-x + 4y + z = 4.$$

15

(c) Find the characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

Hence find A^{-1} and A^6. 15

(d) Define a positive definite quadratic form. Reduce the quadratic form $x_1^2 + x_3^2 + 2x_1x_2 + 2x_2x_3$ to canonical form. Is this quadratic form positive definite ? 15

Q. 3. (a) Let the roots of the equation in λ.

$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$

be u, v, w. Prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}$$ 15

(b) Prove that an equation of the form $x^n = \alpha$, where $n \in N$ and $\alpha > 0$ is a real number, has a positive root. 15

(c) Prove that : $\int \frac{x^2 + y^2}{p} dx = \frac{\pi ab}{4}[4 + (a^2 + b^2)(a^{-2} + b^{-2})]$,

when the integral is taken round the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and p is the length of three perpendicular from the centre to the tangent. 15

(d) If the function f is defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

then show that f possesses both the partial derivatives at (0, 0) but it is not continuous thereat. 15

Q. 4. (a) Find the locus of the middle points of the chords of the rectangular hyperbola $x^2 - y^2 = a^2$ which touch the parabola $y^2 = 4ax$. 15

(b) Prove that the locus of a line which meets the lines $y = \pm mx$, $z = \pm c$ and the circle $x^2 + y^2 = a^2$, $z = 0$ is $c^2 m^2 (cy - mzx)^2 + c^2 (yz - cmx)^2 = a^2 m^2 (z^2 - c^2)^2$. 15

(c) Prove that the lines of intersection of pairs of tangent planes to $ax^2 + by^2 + cz^2 = 0$ which touch along perpendicular generators lie on the cone $a^2 (b + c) x^2 + b^2 (c + a) y^2 + c^2 (a + b) z^2 = 0$. 15

(d) Tangent planes are drawn to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ through the point (α, β, γ). prove that the perpendiculars to them through the origin generate the cone $(\alpha x + \beta y + \gamma z)^2 = a^2 x^2 + b^2 y^2 + c^2 z^2$. 15

SECTION 'B'

Q. 5. Attempt any *five* of the following :

(a) Find the solution of the following differential equation $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$. 12

(b) Solve : $y (xy + 2x^2 y^2)\, dx + x (xy - x^2 y^2)\, dy = 0$. 12

(c) A point moving with uniform acceleration describes distances s_1 and s_2 metres in successive intervals of time t_1 and t_2 seconds. Express the acceleration in terms of s_1, s_2, t_1 and t_2. 12

(d) A non uniform string hangs under gravity. Its cross-section at any point is inversely proportional to the tension at that point. Prove that the curve in which the string hangs is an arc of a parabola with its axis vertical. 12

(e) A circular area of radius a is immersed with its plane vertical, and its centre at a depth c. Find the position of its centre of pressure. 12

(f) Show that if $\overline{A}$ and $\overline{B}$ are irrotational, then $\overline{A} \times \overline{B}$ is solenoidal. 12

Q. 6. (a) Solve : $(D^4 - 4D^2 - 5)\, y = e^x (x + \cos x)$. 15

(b) Reduce the equation $(px - y)(py + x) = 2p$, where $p = \dfrac{dy}{dx}$ to Clairaut's equation and hence solve it. 15

(c) Solve : $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)e^x$. 15

(d) Solve the following differential equation :

$$(1-x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - (1+x^2)y = x.$$ 15

Q. 7. (a) Prove that the velocity required to project a particle from a height h to fall at a horizontal distance a from a point of projection, is at least equal to $\sqrt{g\left[\sqrt{a^2+h^2} - h\right]}$. 15

(b) A car of mass 750 kg is running up a hill of 1 in 30 at a steady speed of 36 km/hr; the friction is equal to the weight of 40 kg. Find the work done in 1 second. 15

(c) A uniform bar AB weights 12 N and rests with one part, AC of length 8 m, on a horizontal table and the remaining part CB projecting over the edge of the table. If the bar is on the point of overbalancing when a weight of 5 N is placed on it at a point 2m from

A and a weight of 7 N is hung from B, find the length of AB. 15

(d) A cone, of given weight and volume, floats with its vertex downwards. Prove that the surface of the cone in contact with the liquid is least when its vertical angle is $2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$. 15

Q. 8. (a) Show that the Frenet-Serret formulae can be written in the form

$$\frac{d\overline{T}}{ds} = \overline{\omega} \times \overline{T}, \frac{d\overline{N}}{ds} = \overline{\omega} \times \overline{N} \text{ and } \frac{d\overline{B}}{dx} = \overline{\omega} \times \overline{B},$$

where $\overline{\omega} = \tau\overline{T} + k\overline{B}$.

15

(b) Prove the identity

$$\nabla(\overline{A} \cdot \overline{B}) = (\overline{B} \cdot \nabla)\overline{A} + (\overline{A} \cdot \nabla)\overline{B} + \overline{B} \times (\nabla \times \overline{A}) + \overline{A} \times (\nabla \times \overline{B}) \cdot$$ 15

(c) Derive the identity

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi).\hat{n}dS,$$

where V is the volume bounded by the closed surface S. 15

(d) Verify Stokes' theorem for

$$\bar{f} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k},$$

where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. 15

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