

MATHEMATICS



Paper I

Time Allowed : Three Hours

Maximum Marks : 200

INSTRUCTIONS

Candidates should attempt Questions No. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section.

All questions carry equal marks.

Marks allotted to parts of a question are indicated against each.

Answers must be written in ENGLISH only.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless indicated otherwise, symbols & notations carry their usual meaning.

SECTION A

1. Answer any *four* of the following :

- (a) Let V be the vector space of 2×2 matrices over the field of real numbers \mathbf{R} . Let $W = \{A \in V \mid \text{Trace } A = 0\}$. Show that W is a subspace of V . Find a basis of W and dimension of W .

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- (b) Find the linear transformation from \mathbf{R}^3 into \mathbf{R}^3 which has its range the subspace spanned by $(1, 0, -1), (1, 2, 2)$. 10

- (c) Show that the function defined by

$$f(x, y) = \begin{cases} \frac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$$

is discontinuous at the origin but possesses partial derivatives f_x and f_y thereat. 10

- (d) Let the function f be defined by

$$f(t) = \begin{cases} 0, & \text{for } t < 0 \\ t, & \text{for } 0 \leq t \leq 1 \\ 4, & \text{for } t > 1. \end{cases}$$

- (i) Determine the function $F(x) = \int_0^x f(t) dt$.

- (ii) Where is F non-differentiable? Justify your answer. 10

- (e) A variable plane is at a constant distance p from the origin and meets the axes at A, B, C . Prove that the locus of the centroid of the tetrahedron

$$\text{OABC is } \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}. \quad 10$$

2. (a) Let $V = \{(x, y, z, u) \in \mathbf{R}^4 : y + z + u = 0\}$,
 $W = \{(x, y, z, u) \in \mathbf{R}^4 : x + y = 0, z = 2u\}$
 be two subspaces of \mathbf{R}^4 . Find bases for V , W ,
 $V + W$ and $V \cap W$. 10

- (b) Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

and hence compute A^{10} . 10

- (c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$.

Find an invertible matrix P such that $P^{-1}AP$ is a
 diagonal matrix. 10

- (d) Find an orthogonal transformation to reduce the
 quadratic form $5x^2 + 2y^2 + 4xy$ to a canonical
 form. 10

3. (a) Show that the equation $3^x + 4^x = 5^x$ has exactly
 one root. 8

- (b) Test for convergence the integral $\int_0^{\infty} \sqrt{x} e^{-x} dx$. 8

- (c) Show that the area of the surface of the sphere
 $x^2 + y^2 + z^2 = a^2$ cut off by $x^2 + y^2 = ax$ is
 $2(\pi - 2)a^2$. 12

(d) Show that the function defined by

$$f(x, y, z) = 3 \log (x^2 + y^2 + z^2) - 2x^2 - 2y^3 - 2z^3,$$
$$(x, y, z) \neq (0, 0, 0)$$

has only one extreme value, $\log \left(\frac{3}{e^2} \right)$.

4. (a) Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$.

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(b) Find the tangent planes to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which are parallel to the plane $lx + my + nz = 0$.

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(c) Prove that the semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord.

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(d) Tangent planes at two points P and Q of a paraboloid meet in the line RS. Show that the plane through RS and middle point of PQ is parallel to the axis of the paraboloid.

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SECTION B

5. Answer any *four* of the following :

(a) Find the family of curves whose tangents form an angle $\pi/4$ with hyperbolas $xy = c$. 10

(b) Solve :

$$\frac{d^2y}{dx^2} - 2 \tan x \frac{dy}{dx} + 5y = \sec x \cdot e^x. \quad 10$$

(c) The apses of a satellite of the Earth are at distances r_1 and r_2 from the centre of the Earth. Find the velocities at the apses in terms of r_1 and r_2 . 10

(d) A cable of length 160 meters and weighing 2 kg per meter is suspended from two points in the same horizontal plane. The tension at the points of support is 200 kg. Show that the span of the cable is $120 \cosh^{-1} \left(\frac{5}{3} \right)$ and also find the sag. 10

(e) Evaluate the line integral

$$\oint_C (\sin x \, dx + y^2 \, dy - dz), \text{ where } C \text{ is the circle } x^2 + y^2 = 16, z = 3, \text{ by using Stokes' theorem.} \quad 10$$

6. (a) Solve :

$$p^2 + 2py \cot x = y^2,$$

$$\text{where } p = \frac{dy}{dx}. \quad 10$$

(b) Solve :

$$\{x^4 D^4 + 6x^3 D^3 + 9x^2 D^2 + 3x D + 1\}y = (1 + \log x)^2,$$

$$\text{where } D \equiv \frac{d}{dx}. \quad 15$$

(c) Solve :

$$(D^4 + D^2 + 1)y = ax^2 + be^{-x} \sin 2x,$$

$$\text{where } D \equiv \frac{d}{dx}.$$

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7. (a) One end of a uniform rod AB, of length $2a$ and weight W , is attached by a frictionless joint to a smooth wall and the other end B is smoothly hinged to an equal rod BC. The middle points of the rods are connected by an elastic cord of natural length a and modulus of elasticity $4W$. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rod is $2 \sin^{-1} \left(\frac{3}{4} \right)$.

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- (b) AB is a uniform rod, of length $8a$, which can turn freely about the end A, which is fixed. C is a smooth ring, whose weight is twice that of the rod, which can slide on the rod, and is attached by a string CD to a point D in the same horizontal plane as the point A. If AD and CD are each of length a , fix the position of the ring and the tension of the string when the system is in equilibrium.

Show also that the action on the rod at the fixed end A is a horizontal force equal to $\sqrt{3} W$, where W is the weight of the rod.

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- (c) A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d ; if V and v be the corresponding velocities of the stream and if the motion is supposed to be that of the divergence from the vertex of cone, prove that

$$\frac{v}{V} = \frac{D^2}{d^2} e^{(v^2 - V^2)/2K}$$

where K is the pressure divided by the density and supposed constant.

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8. (a) Find the curvature, torsion and the relation between the arc length S and parameter u for the curve :

$$\vec{r} = \vec{r}(u) = 2 \log_e u \hat{i} + 4u \hat{j} + (2u^2 + 1) \hat{k} \quad 10$$

- (b) Prove the vector identity :

$$\text{curl}(\vec{f} \times \vec{g}) = \vec{f} \text{div} \vec{g} - \vec{g} \text{div} \vec{f} + (\vec{g} \cdot \nabla) \vec{f} - (\vec{f} \cdot \nabla) \vec{g}$$

and verify it for the vectors $\vec{f} = x \hat{i} + z \hat{j} + y \hat{k}$

and $\vec{g} = y \hat{i} + z \hat{k}$.

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- (c) Verify Green's theorem in the plane for

$$\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy],$$

where C is the boundary of the region enclosed by the curves $y = \sqrt{x}$ and $y = x^2$.

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- (d) The position vector \vec{r} of a particle of mass 2 units at any time t , referred to fixed origin and axes, is

$$\vec{r} = (t^2 - 2t)\hat{i} + \left(\frac{1}{2}t^2 + 1\right)\hat{j} + \frac{1}{2}t^2\hat{k}.$$

At time $t = 1$, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin.

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