005412

B-JGT-K-NBA

MATHEMATICS

IMS™ (INSTITUTE OF MATHEMATICAL SCIENCES)

Paper I

Time Allowed : Three Hours | *Maximum Marks : 200*

INSTRUCTIONS

Candidates should attempt questions 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section.

All questions carry equal marks. Marks allotted to parts of a question are indicated against each.

Answers must be written in ENGLISH only.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless indicated otherwise, symbols & notations carry their usual meaning.

SECTION A

1. Answer any *five* of the following :

(a) Show that the set

$$P[t] = \{at^2 + bt + c \,/\, a, b, c \in \mathbb{R}\}$$

forms a vector space over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space ? *8*

(b) Determine whether the quadratic form

$$q = x^2 + y^2 + 2xz + 4yz + 3z^2$$

is positive definite. *8*

(c) Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$. *8*

(d) Let f be a function defined on $\mathbb{R}$ such that

$$f(x + y) = f(x) + f(y), \quad x, y \in \mathbb{R}.$$

If f is differentiable at one point of $\mathbb{R}$, then prove that f is differentiable on $\mathbb{R}$. *8*

(e) If a plane cuts the axes in A, B, C and (a, b, c) are the coordinates of the centroid of the triangle ABC, then show that the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$. *8*

(f) Find the equations of the spheres passing through the circle

$$x^2 + y^2 + z^2 - 6x - 2z + 5 = 0, \quad y = 0$$

and touching the plane $3y + 4z + 5 = 0$. *8*

2\. (a) Show that the vectors

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 2, 1), \quad \alpha_3 = (0, -3, 2)$$

form a basis for $\mathbb{R}^3$. Find the components of (1, 0, 0) w.r.t. the basis $\{\alpha_1, \alpha_2, \alpha_3\}$. *10*

(b) Find the characteristic polynomial of $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Verify Cayley – Hamilton theorem for this matrix and hence find its inverse. 10

(c) Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. 12

(d) Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{pmatrix}$$

8

3. (a) Discuss the convergence of the integral

$$\int_0^\infty \frac{dx}{1+x^4 \sin^2 x}$$

10

(b) Find the extreme value of xyz if $x + y + z = a$. 10

(c) Let

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that :

(i) $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

(ii) f is differentiable at (0, 0) *10*

(d) Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$. *10*

4. (a) Prove that the second degree equation

$$x^2 - 2y^2 + 3z^2 + 5yz - 6zx - 4xy + 8x - 19y - 2z - 20 = 0$$

represents a cone whose vertex is (1, –2, 3). *10*

(b) If the feet of three normals drawn from a point P to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lie in the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, prove that the feet of the other three normals lie in the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + 1 = 0.$$ *10*

(c) If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equations of the other two. 10

(d) Prove that the locus of the point of intersection of three tangent planes to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, which are parallel to the conjugate diametral planes of the ellipsoid $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$ is

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}$$ 10

SECTION B

5. Answer any ***five*** of the following :

(a) Show that $\cos(x + y)$ is an integrating factor of

$$y\,dx + [y + \tan(x + y)]\,dy = 0.$$

Hence solve it. *8*

(b) Solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$$ *8*

(c) A uniform rod AB rests with one end on a smooth vertical wall and the other on a smooth inclined plane, making an angle α with the horizon. Find the positions of equilibrium and discuss stability. *8*

(d) A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If θ_1 and θ_2 be the base angles and θ be the angle of projection, prove that,

$$\tan\theta = \tan\theta_1 + \tan\theta_2.$$ *8*

(e) Prove that the horizontal line through the centre of pressure of a rectangle immersed in a liquid with one side in the surface, divides the rectangle in two parts, the fluid pressure on which, are in the ratio, 4 : 5. *8*

(f) Find the directional derivation of $\vec{V}^2$, where, $\vec{V} = xy^2\vec{i} + zy^2\vec{j} + xz^2\vec{k}$ at the point (2, 0, 3) in the direction of the outward normal to the surface $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1). 8

6. (a) Solve the following differential equation

$$\frac{dy}{dx} = \sin^2(x - y + 6)$$ 8

(b) Find the general solution of

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 + 1)y = 0$$ 12

(c) Solve

$$\left(\frac{d}{dx} - 1\right)^2\left(\frac{d^2}{dx^2} + 1\right)^2 y = x + e^x$$ 10

(d) Solve by the method of variation of parameters the following equation

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = (x^2 - 1)^2$$ 10

7. (a) A uniform chain of length $2l$ and weight W, is suspended from two points A and B in the same horizontal line. A load P is now hung from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is,

$$\frac{1}{2}\left[P.\frac{l}{h} + W.\frac{h^2 + l^2}{2hl}\right].$$ 14

(b) A particle moves with a central acceleration $\frac{\mu}{(\text{distance})^2}$, it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is,

$$\sin^{-1}\left[\frac{\mu}{VR\left(V^2 - \frac{2\mu}{R}\right)^{1/2}}\right].$$

13

(c) A smooth wedge of mass M is placed on a smooth horizontal plane and a particle of mass m slides down its slant face which is inclined at an angle α to the horizontal plane. Prove that the acceleration of the wedge is,

$$\frac{mg \sin\alpha \cos\alpha}{M + m \sin^2\alpha}.$$

13

8\. (a) (i) Show that

$$\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3z^2x\vec{k}$$

is a conservative field. Find its scalar potential and also the work done in moving a particle from (1, –2, 1) to (3, 1, 4). 5

(ii) Show that, $\nabla^2 f(r) = \left(\frac{2}{r}\right)f'(r) + f''(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. 5

(b) Use divergence theorem to evaluate,

$$\iint_S (x^3 \, dy \, dz + x^2 y \, dz \, dx + x^2 z \, dy \, dx),$$

where S is the sphere, $x^2 + y^2 + z^2 = 1$. *10*

(c) If $\vec{A} = 2y\,\vec{i} - z\,\vec{j} - x^2\,\vec{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$, $z = 6$, evaluate the surface integral,

$$\iint_S \vec{A} \cdot \hat{n} \, \overrightarrow{dS}.$$

10

(d) Use Green's theorem in a plane to evaluate the integral, $\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, where C is the boundary of the surface in the xy-plane enclosed by, $y = 0$ and the semi-circle, $y = \sqrt{1-x^2}$. *10*