MATHEMATICS

Paper – I

Time Allowed : ***Three Hours***

Maximum Marks : ***200***

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

There are ***EIGHT*** *questions in all, out of which* ***FIVE*** *are to be attempted.*

Questions no. ***1*** *and* ***5*** *are* ***compulsory****. Out of the remaining* ***SIX*** *questions,* ***THREE*** *are to be attempted selecting at least* ***ONE*** *question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Answers must be written in ***ENGLISH*** *only.*

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

SECTION A

Q1. (a) Let A be a square matrix of order 3 such that each of its diagonal elements is 'a' and each of its off-diagonal elements is 1. If B = bA is orthogonal, determine the values of a and b. *8*

(b) Let V be the vector space of all 2×2 matrices over the field R. Show that W is not a subspace of V, where

(i) W contains all 2×2 matrices with zero determinant.

(ii) W consists of all 2×2 matrices A such that $A^2 = A$. *8*

(c) Using the Mean Value Theorem, show that

(i) f(x) is constant in [a, b], if $f'(x) = 0$ in [a, b].

(ii) f(x) is a decreasing function in (a, b), if f′(x) exists and is < 0 everywhere in (a, b). 8

(d) Let $u(x, y) = ax^2 + 2hxy + by^2$ and $v(x, y) = Ax^2 + 2Hxy + By^2$. Find the Jacobian $J = \frac{\partial(u, v)}{\partial(x, y)}$, and hence show that u, v are independent unless $\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$. *8*

(e) Find the equations of the planes parallel to the plane $3x - 2y + 6z + 8 = 0$ and at a distance 2 from it. *8*

Q2. (a) State the Cayley-Hamilton theorem. Verify this theorem for the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \text{ Hence find } A^{-1}.$$ *10*

(b) Show that

$$\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, \; p, q > -1.$$

Hence evaluate the following integrals :

(i) $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$

(ii) $\int_0^1 x^3(1-x^2)^{5/2} \, dx$

(iii) $\int_0^1 x^4(1-x)^3 \, dx$ 10

(c) Find the maxima and minima for the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$$

Also find the saddle points (if any) for the function. 10

(d) Show that the angles between the planes given by the equation $2x^2 - y^2 + 3z^2 - xy + 7zx + 2yz = 0$ is $\tan^{-1} \frac{\sqrt{50}}{4}$. 10

Q3. (a) Reduce the following matrix to a row-reduced echelon form and hence find its rank : 10

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

(b) Given that the set {u, v, w} is linearly independent, examine the sets

(i) {u + v, v + w, w + u}

(ii) {u + v, u − v, u − 2v + 2w}

for linear independence.

(c) Evaluate the integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$, by changing to polar coordinates. Hence show that $\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$. 10

(d) Find the angle between the lines whose direction cosines are given by the relations $l + m + n = 0$ and $2lm + 2ln - mn = 0$. 10

Q4. (a) Find the eigenvalues and the corresponding eigenvectors for the matrix $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$. Examine whether the matrix A is diagonalizable. Obtain a matrix D (if it is diagonalizable) such that $D = P^{-1} A P$. *10*

(b) A function f(x, y) is defined as follows :

$$f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}.$$

Show that $f_{xy}(0, 0) = f_{yx}(0, 0)$. *10*

(c) Find the equation of the right circular cone with vertex at the origin and whose axis makes equal angles with the coordinate axes and the generator is the line passing through the origin with direction ratios $(1, -2, 2)$. *10*

(d) Find the shortest distance and the equation of the line of the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$. *10*

SECTION B

Q5. (a) Solve

$$(2D^3 - 7D^2 + 7D - 2)\, y = e^{-8x} \text{ where } D = \frac{d}{dx}.$$ 8

(b) Solve the differential equation

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4.$$ 8

(c) A particle is undergoing simple harmonic motion of period T about a centre O and it passes through the position P (OP = b) with velocity v in the direction OP. Prove that the time that elapses before it returns to P is $\frac{T}{\pi} \tan^{-1}\left(\frac{vT}{2\pi b}\right)$. 8

(d) A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\frac{\pi r}{2}$, then prove that the total angle through which the cube can swing without falling is 90°. 8

(e) Prove that

$$\nabla^2 r^n = n\,(n + 1)\, r^{n-2}$$

and that $r^n \vec{r}$ is irrotational, where $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. 8

Q6. (a) Solve the differential equation

$$\left(\frac{dy}{dx}\right)^2 + 2 \cdot \frac{dy}{dx} \cdot y \cot x = y^2.$$ 15

(b) A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods is supported in a horizontal position, then prove that the tension of the string is $\frac{2W(2b^2 - a^2)}{b\sqrt{4b^2 - a^2}}$. 10

(c) Using Stokes' theorem, evaluate

$$\oint_C [(x + y)\,dx + (2x - z)\,dy + (y + z)\,dz],$$

where C is the boundary of the triangle with vertices at (2, 0, 0), (0, 3, 0) and (0, 0, 6). 15

Q7. (a) Solve the differential equation

$$e^{3x}\left(\frac{dy}{dx} - 1\right) + \left(\frac{dy}{dx}\right)^3 e^{2y} = 0.$$ 10

(b) A planet is describing an ellipse about the Sun as a focus. Show that its velocity away from the Sun is the greatest when the radius vector to the planet is at a right angle to the major axis of path and that the velocity then is $\frac{2\pi ae}{T\sqrt{1-e^2}}$, where 2a is the major axis, e is the eccentricity and T is the periodic time. 10

(c) A semi-ellipse bounded by its minor axis is just immersed in a liquid, the density of which varies as the depth. If the minor axis lies on the surface, then find the eccentricity in order that the focus may be the centre of pressure. 10

(d) Evaluate

$$\iint_S (\nabla \times \vec{f}\,) \cdot \hat{n}\, dS,$$

where S is the surface of the cone, $z = 2 - \sqrt{x^2 + y^2}$ above xy-plane and $\vec{f} = (x - z)\hat{i} + (x^3 + yz)\hat{j} - 3xy^2\hat{k}$. 10

Q8. (a) Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$ by using the method of variation of parameter. *10*

(b) A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu\left(\frac{a^5}{x^2}\right)^{\frac{1}{3}}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then prove that it will arrive at O with a velocity $a\sqrt{6\mu}$ after time $\frac{8}{15}\sqrt{\frac{6}{\mu}}$. *10*

(c) Find the curvature and torsion of the circular helix

$$\vec{r} = a(\cos\theta, \sin\theta, \theta\cot\beta),$$

β is the constant angle at which it cuts its generators. *10*

(d) If the tangent to a curve makes a constant angle α, with a fixed line, then prove that $\kappa\cos\alpha \pm \tau\sin\alpha = 0$.

Conversely, if $\frac{\kappa}{\tau}$ is constant, then show that the tangent makes a constant angle with a fixed direction. *10*

विप्रोज्य **DETACHABLE**

MATHEMATICS

Paper II

Time Allowed : Three Hours	*Maximum Marks : 200*

QUESTION PAPER SPECIFIC INSTRUCTIONS

Please read each of the following instructions carefully before attempting questions.

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Question Nos. **1** and **5** are compulsory. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two **Sections A** and **B**.

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SECTION 'A'

1. (a) Prove that every group of order four is Abelian. 8

1. (b) A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as below :

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$$

Prove that f is continuous at $x = \frac{1}{2}$ but discontinuous at all other points in $\mathbb{R}$. 10

1.(c) If $f(z) = u(x, y) + iv(x, y)$ is an analytic function of $z = x + iy$ and $u + 2v = x^3 - 2y^3 + 3xy(2x - y)$ then find $f(z)$ in terms of z. 8

1.(d) Solve by simplex method the following LPP :

Minimize $Z = x_1 - 3x_2 + 2x_3$

subject to the constraints

$3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 0$

and $x_1, x_2, x_3 \geq 0$ 14

2.(a) Let G be the set of all real numbers except -1 and define $a*b = a + b + ab$ $\forall\ a, b \in G$. Examine if G is an Abelian group under $*$. 10

2.(b) Let H and K are two finite normal subgroups of co-prime order of a group G. Prove that $hk = kh\ \forall h \in H$ and $k \in K$. 10

2.(c) Let A be an ideal of a commutative ring R and $B = \{x \in R : x^n \in A$ for some positive integer $n\}$. Is B an ideal of R ? Justify your answer. 10

2.(d) Prove that the ring $\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}, i = \sqrt{-1}\}$ of Gaussian integers is a Euclidean domain. 10

3.(a) Evaluate $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ given that

$$f(x, y) = \begin{cases} x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y} & \text{if } xy \neq 0 \\ 0 & \text{, otherwise} \end{cases}$$ 10

3.(b) Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the condition $\dfrac{x^2}{4} + \dfrac{y^2}{5} + \dfrac{z^2}{25} = 1$. 10

3.(c) Prove that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent but not absolutely convergent. 12

3.(d) Find the volume of the region common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. 8

4.(a) Prove by the method of contour integration that $\displaystyle\int_0^\pi \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\,d\theta = 0$. 12

4.(b) Find the sum of residues of $f(z) = \dfrac{\sin z}{\cos z}$ at its poles inside the circle $|z| = 2$. 8

4.(c) Evaluate $\int_{x=0}^{\infty} \int_{y=0}^{x} x\, e^{-x^2/y}\, dy\, dx$ 8

4.(d) A computer centre has four expert programmers. The centre needs four application programs to be developed. The head of the centre after studying carefully the programs to be developed, estimates the computer times in hours required by the experts to the application programs as follows :

Programs

Programmer	A	B	C	D
P_1	5	3	2	8
P_2	7	9	2	6
P_3	6	4	5	7
P_4	5	7	7	8

Assign the programs to the programmers in such a way that total computer time is least. 12

SECTION 'B'

5.(a) Form the partial differential equation by eliminating arbitrary functions φ and ψ from the relation $z = \varphi(x^2 - y) + \psi(x^2 + y)$. 8

5.(b) Write a BASIC program to compute the multiplicative inverse of a non-singular square matrix. 12

5.(c) A uniform rectangular parallelopiped of mass M has edges of lengths $2a$, $2b$, $2c$. Find the moment of inertia of this rectangular parallelopiped about the line through its centre parallel to the edge of length $2a$. 10

5.(d) Evaluate $\int_0^1 e^{-x^2} dx$ using the composite trapezoidal rule with four decimal precision, i.e., with the absolute value of the error not exceeding 5×10^{-5}. 10

6.(a) Solve the partial differential equation :

$$(x-y)\frac{\partial z}{\partial x} + (x+y)\frac{\partial z}{\partial y} = 2xz$$ 8

6.(b) Find the surface which is orthogonal to the family of surfaces $z(x+y) = c(3z+1)$ and which passes through the circle $x^2 + y^2 = 1$, $z = 1$. 8

6.(c) Find complete integral of $xp - yq = xqf(z - px - qy)$ where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. 12

6.(d) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. It is released from rest from this position, find the displacement $y(x, t)$. 12

7. (a) Find the real root of the equation $x^3 + x^2 + 3x + 4 = 0$ correct up to five places of decimal using Newton-Raphson method. 10

7. (b) A river is 80 metre wide, the depth y, in metre, of the river at a distance x from one bank is given by the following table :

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find the area of cross-section of the river using Simpson's $\frac{1}{3}$rd rule. 10

7. (c) Find y for $x = 0{\cdot}2$ taking $h = 0{\cdot}1$ by modified Euler's method and compute the error, given that : $\frac{dy}{dx} = x + y,\ y(0) = 1$. 10

7. (d) Assuming a 32 bit computer representation of signed integers using 2's complement representation, add the two numbers -1 and -1024 and give the answer in 2's complement representation. 10

8. (a) Consider a mass m on the end of a spring of natural length l and spring constant k. Let y be the vertical coordinate of the mass as measured from the top of the spring. Assume that the mass can only move up and down in the vertical direction. Show that

$$L = \tfrac{1}{2} my'^2 - \tfrac{1}{2} k(y-l)^2 + mgy$$

Also determine and solve the corresponding Euler-Lagrange equations of motion. 12

8. (b) Find the streamlines and pathlines of the two dimensional velocity field :

$u = \frac{x}{1+t},\ v = y,\ w = 0.$

8. (c) The velocity vector in the flow field is given by

$\vec{q} = (az - by)\hat{i} + (bx - cz)\hat{j} + (cy - ax)\hat{k}$

where a, b, c are non-zero constants. Determine the equations of vortex lines. 8

8. (d) Solve Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions

$u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, a) = \sin\left(\frac{n\pi x}{l}\right)$. 12