# MATHEMATICS

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

## PAPER - I

## SECTION A

1. Answer any four of the following :

(a) Suppose U and W are subspaces of the vector space $R^4$ (R) generated by the sets
$B_1 = \{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\}$
$B_2 = \{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}$
respectively. Determine dim (U+W).

(10)

(b) Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

And verify that it satisfies by A.

(10)

(c) If a function $f$ is such that its derivative $f'$ is continuous on [a, b] and derivable on ]a, b [, then show that there exists a number $c$ between $a$ and $b$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(c).$$

(d) If

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^4}, & (x, y) \neq (0,0) \\ 0 \quad , & (x, y) = (0,0) \end{cases}$$

Show that both the partial derivatives exist at (0,0) but the function is not continuous thereat.

(10)

(e) If the three concurrent lines whose direction cosines are $(l_1, m_1, n_1)$, $(l_2, m_2, n_2)$, $(l_3, m_3, n_3)$ are coplanar, prove that

$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = 0$$

(10)

2. (a) Show that the solutions of the differential equation

$$2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$$

is a subspace of the vector space of all real valued continuous functions.

(10)

(b) how that vectors (0, 2, —4), (1, —2, —1), 1, —4, 3) are linearly dependent. Also express

(0, 2, —4) as a linear combination of(l. —2, -1) and 1, —4, 3).

(c) Is the matrix

$$A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$$

similar over the field R to a diagonal matrix? Is A similar over the field C to a diagonal matrix?

(10)

(d) Determine the definiteness of the following quadratic form :

$$q(x_1, x_2, x_3) = [x_1\ x_2\ x_3] \begin{bmatrix} 2 & 0 & -1 \\ 1 & 5 & 2 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(10)

3. (a) Find the values of a and b, so that

$$\operatorname*{Lt}_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} = 1.$$

What are these conditions?

(10)

(b) Show that $f(xy, z - 2x) = 0$

satisfies, under certain conditions, the equation

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x.$$

What are these conditions?

(10)

(c) Find the surface area generated by the revolution of the cardioids $r = a\ (1 + \cos\ \theta)$ about the initial line.

(10)

(d) The function $f$ is defined on ]o, 1[ by

$$f(x) = (-1)^{n+1} n(n+1),\ \frac{1}{n+1} \le x \le \frac{1}{n},\ n \in N.$$

Show that

$$\int_0^1 f(x)\ dx$$

Does not converge.

(10)

4. (a) Find the equations of the three planes through the line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

Parallel to the axes.

(b) Prove that the shortest distance between the line

$z = x \tan\theta,\ y = 0$

and any tangent to the ellipse

$x^2 \sin^2\theta + y^2 = a^2,\ z = 0$

is constant in length.

(10)

(c) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cut the axes in A,B,C. Find the equation of the cone whose vertex is origin and the guiding curve is the circle ABC.

(10)

(d) Find the equation of the cylinder generated $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$, the guiding curve being the conic $z = 2, 3x^2 + 4xy + 5y^2 = 1$.

(10)

## SECTION B

5. Answer any four of the following:

(a) Find the orthogonal trajectories of the family of the curves

$$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1,\ \lambda \text{ being a parameter.}$$

(10)

(b) Show that $e^{2x}$ and $e^{3x}$ are linearly independent solutions of

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

Find the general solution when $y(0) = 0$ and

$$\left.\frac{dy}{dx}\right]_0 = 1.$$

(10)

(c) AB, BC are two equal, similar rods freely hinged at B and lie, in a straight line on a smooth table. The end A is struck by a blow perpendicular to AB. Show that the resulting velocity of A is $3\frac{1}{2}$ times of B.

(d) A triangle ABC is immersed in a liquid with the vertex C in the surface, and the sides AC, BC equally inclined to the surface. Show that the vertical through C divides the triangle into two others, the fluid pressure upon which are as

$b^3 + 3ab^2 : a^3 + 3a^2b.$

(10)

(e) Evaluate

$$\int_C \vec{F}\cdot d\vec{r}$$

Where $\vec{F} = c[-3a\sin^2\theta\cos\theta\,\vec{i} + a(2\sin\theta - 3\sin^2\theta)\,\vec{j} + b\sin 2\theta\,\vec{k}]$ and the curve C is given by

$\vec{r} = a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j} + b\theta\,\vec{k}$

$\theta$ varying from $\frac{\pi}{4} to \frac{\pi}{2}$

(10)

6. (a) Find the family of curves whose tangents form an $\frac{\pi}{4}$ angle with the hyperbola $xy = C$.

(10)

(b) Apply. the method of variation of parameters to solve

$(D^2 + a^2)y = \operatorname{cosec} ax.$

(c) Solve

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4}y = 0$$

By using the method of removal of first derivate.

(10)

(d) Find the general solution of

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 3y = 0, \text{if } y = x \text{ is a}$$

Solution of it.

(10)

7. (a) A right angled triangular prism floats in a fluid of which the density varies as the depth with the right angle immersed and the edges horizontal. Show that the curve buoyancy is of the form

$r^3 \sin^2\theta \cos^2\theta = c^3$

(14)

(b) A heavy chain of length $2l$ has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be $n$ times the weight of the chain, show that its greatest possible distance from A is

$$\frac{2l}{\lambda}\log\left\{\lambda + \sqrt{(1+\lambda^2)}\right\},$$

Where

$\frac{1}{\lambda} = \mu(2n+1), \mu$ being the coefficient of friction.

(13)

(c) Two like rods AB and BC, each of length $2a$ are freely jointed at B; AB can turn round the end A, and C can move freely on a vertical straight line through A and they are then released. Initially the rods are held in a horizontal line, C being in coincidence with A and they are then released. Show that when the rods are inclined at an angle $\theta$ to the horizontal, the angular velocity of either is

$$\sqrt{\left(\frac{3g}{a}\cdot\frac{\sin\theta}{1+3\cos^2\theta}\right)}$$

(13)

8. (a) Show that

$$\text{Curl}\quad\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^3}\left(\bar{a}.\bar{r}\right)$$

Where $\vec{a}$ is a constant vector and

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

(10)

(b) Find the curvature and torsion at any point of the curve

$$x = a \cos 2t,\ y = a \sin 2t,\ z = 2a \sin t.$$

(10)

(c) Evaluate the surface integral

$$\int_S \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right) . d\vec{a}$$

Where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

(10)

(d) Apply Stokes' theorem to prove that

$$\int_n (y\,dx + z\,dy + x\,dz) = -2\sqrt{2}\pi a^2,$$

Where C is curve given by

$$x^2 + y^2 + z^2 - 2ax - 2ay = 0,\ x + y = 2a$$

(10)

# MATHEMATICS

## PAPER - II

## SECTION A

1. Attempt any four parts:

(a) (i) Prove or disprove that if H is a normal subgroup of a group G such that H and G/H are cyclic, then G is cyclic.

(5)

(ii) Show by counter-example that the distributive laws in the definition of a ring is not redundant.

(5)

(b) (i) In the ring of integers modulo 10 (*i.e.*, $Z_{10}, \oplus_{10}, \Theta_{10}$) find the subfields.

(5)

(ii) Prove or disprove that only non-singular matrices form a group under matrix multiplication.

(5)

(c) Show that the series

$$\sum (-1)^n \left[\sqrt{n^2+1} - n\right]$$

Is conditionally convergent.

(10)

(d) If $f(z) = u + iv$ is analytic and

$y = e^{-x} (x \sin y - y \cos y)$

then find $v$ and $f(z)$.

(10)

(e) Solve the following LPP by graphical method:

Maximize $Z = 5x_1 + 7x_2$

subject to

$$\begin{aligned} x_1 + x_2 &\le 4 \\ 3x_1 + 8x_2 &\le 24 \\ 10x_1 + 7x_2 &\le 35 \\ x_1, x_2 &\ge 0. \end{aligned}$$

2. (a) Applying Cauchy's criterion for convergence, show that the sequence $(s_n)$ defined by

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

Is not convergent.

(13)

(b) Expand

$$f(z) = \frac{1}{(z+1)(z+3)}$$

In a Laurent series valid for –

(i) $1 < |z| < 3$

(ii) $|z| > 3$.

(13)

(c) Show that there are no simple groups of order 63 and 56.

(14)

3. (a) Prove that every Euclidean domain is PID.

(14)

(b) Show that

$$\iint_D \frac{(x-y)}{(x+y)^3}\,dx\,dy$$

Does not exist, where

$D = \{(x, y) \in R^2 \mid 0 \le x \le 1, 0 \le y \le 1\}$

(c) Solve the following LPP by simplex method:

(13)

Maximize Z = $2x_1 + 5x_2 + 7x_3$

Subject to

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &\le 100 \\ x_1 + 4x_2 + 2x_3 &\le 100 \\ x_1 + x_2 + 3x_3 &\le 100 \\ x_1, x_2, x_3 &\ge 0. \end{aligned}$$

4. (a) If $f : R^2 \to R$ such that

$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Then show that $f_{xy} \ne f_{yx}$.

(13)

(b) Using residue theorem, evaluate

$$\int_0^{2\pi} \frac{d\theta}{(3 - 2\cos\theta + \sin\theta)}.$$

(14)

(c) Slow the following minimal assignment problem:

| *Man* → | *1* | *2* | *3* | *4* |
|---|---|---|---|---|
| *I* | 12 | 30 | 21 | 15 |
| *II* | 18 | 33 | 9 | 31 |
| *III* | 44 | 25 | 24 | 21 |
| *IV* | 23 | 30 | 28 | 14 |

# SECTION B

5. Attempt any four parts.

(a) Find the smallest positive root of equation $3x + \sin x - e^x = 0$, correct to five decimal places, using Regula-falsi method.

(10)

(b) Find the integral curves of the equations

$$\frac{dx}{(x+z)} = \frac{dy}{y} = \frac{dz}{(z+y^2)}$$

(10)

(c) (i) Multiply $1.01_2$ with $10.1_2$

(ii) Draw a diagram of digital circuit for the function
F(X, Y, Z) YZ + XZ using NAND gate only.

(10)

(d) Find the velocity at any point due to a number of straight parallel vortex filaments in an infinitely extended mass of homogeneous liquid.

(I0)

(e) Show that moment of inertia of the area bounded by $r^2 = a^2 \cos 2\theta$ about its axis is

$$\frac{Ma^2}{16}\left(\pi - \frac{8}{3}\right)$$

(10)

6. (a) Solve

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$$

(13)

(b) Write a computer program using BASIC' to solve the following problem

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{x} dx$$

By trapezoidal rule.

(14)

(c) Derive three-point Gaussian quadrature formula and hence evaluate

$$\int_{0.2}^{1.5} e^{-x^2} dx$$

calculating weights and residues. Give the result to three decimal places.

(13)

7. (a) Show that $\phi = (x-t)(y-t)$ represents the velocity potential of an incompressible two-dimensional fluid. Show that streamlines at time tare the curves

$(x-t)^2(y-t)^2 \neq$ constant

And that the paths of the fluid particles have the equation

$\log(x-y) = \frac{1}{2}\{(x+y) - a(x-y)^{-1}\} + b$

where a and b are constants.

(14)

(b) Find the complete integral of

$p^2x + q^2y = z.$

(13)

(c) Compute y(10) using Lagrange's interpolation formula from the following data:

| | | | | |
|---|---|---|---|---|
| x | 3 | 7 | 11 | 17 |
| y | 10 | 15 | 17 | 20 |

(13)

8 (a) A plank, of mass T, is initially at rest along a line of greatest slope of a smooth plane inclined at an angle $\alpha$ to the horizon, and a man, of mass M', starting from the tipper end walks down the plank so that it does not move; show that he gets to the other end in time

$$\left(\frac{2M'a}{(M+M')g\sin\alpha}\right)^{1/2}$$

(b) Solve the system

$$1.2x_1 + 21.2x_2 + 1.5x_3 + 2.5x_4 = 27.46$$
$$0.9x_1 + 2.5x_2 + 1.3x_3 + 32.1x_4 = 49.72$$
$$2.1x_1 + 1.5x_2 + 19.8x_3 + 1.3x_4 = 28.76$$
$$20.9x_1 + 1.2x_2 + 2.1x_3 + 0.9x_4 = 21.70$$

using Gauss-Seidel iterative scheme correct to three decimal places starting with initial value $(1.04\ 1.30\ 1.45\ 155)^T$

(13)

(c) Two sources, each of strength $m$, are placed at the points $(-a, 0)$, $(a, 0)$ and a sink of strength 2m at the origin. Show that the streamlines are the curves

$$(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$$

Where $\lambda$ is the variable parameter.

(13)