

MATHEMATICS

PAPER - I SECTION A



1. Answer any four of the following:

- (a) Let $V = P_3(\mathbb{R})$ be the vector space of polynomial functions over real of degree- at most 3. Let $D : V \rightarrow V$ be the differentiation operator defined by

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2, \quad x \in \mathbb{R}.$$

- (i) Show that $B = \{1, x, x^2, x^3\}$ is a basis for V .
- (ii) Find the matrix $[D]_B$ with respect to B of D .
- (iii) Show that $B' = \{1, (x+1), (x+1)^2, (x+1)^3\}$ is a basis for V .
- (iv) Find the matrix $[D]_{B'}$ with respect to B' of D .
- (v) Find the matrix $[D]_{B', B}$ of D relative to B' and B .

(10)

- (b) Find the eigen values and the corresponding eigenvectors of $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

(10)

- (c) Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

Show that f is differentiable at each point of reals but $f'(x)$ is not continuous at $x = 0$.

(10)

- (d) Show that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = 2x^2 - 6xy + 3y^2$ has a critical point at $(0, 0)$ and that it is a saddle point.

(10)

- (e) Find the equations of the generators of the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ through any point of the principal elliptic section $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1; z = 0$.

(10)

2. (a) Show that the vectors $v_1 = (1, 1, 1)$, $v_2 = (0, 1, 1)$, $v_3 = (0, 0, 1)$ form a basis for $\mathbb{R}^{(3)}$. Express $v = (3, 1, -4)$ as a linear combination of v_1 , v_2 and v_3 . Is the set $S = \{v, v_1, v_2, v_3\}$ linearly independent?

(10)

- (b) Determine a non-singular matrix P such that $P^t A P$ is a diagonal matrix, where P^t denotes the transpose of P , and $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$.

(10)

- (b) Show that the real quadratic form $\phi = n(x_1^2 + x_2^2 + \dots + x_n^2) - (x_1 + x_2 + \dots + x_n)^2$ in n variables is positive semi-definite.

(10)

3. (a) (i) Using Taylor's theorem with remainder show that

$$x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120} \text{ for all } x \geq 0.$$

(5)

- (ii) Let $f: \mathbf{R}^2 \rightarrow \mathbf{R}$ be defined by

$$f(x, y) = \frac{xy}{x^2 - y^2} \text{ if } x \neq \pm y$$

$$= 0 \text{ if } x = \pm y.$$

Show that $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

(5)

- (b) Show that the curve given by

$$x^3 - 4x^2y + 5xy^2 - 2y^3 + 3x^2 - 4xy + 2y^2 - 3x + 2y - 1 = 0$$

has only one asymptote given by

$$y = \frac{1}{2}x + 3.$$

(10)

- (c) Find the extremum values of

$$f(x, y) = 2x^2 - 8xy + 9y^2 \text{ on } x^2 + y^2 - 1 = 0$$

using Lagrange multiplier method.

(10)

- (d) A solid cuboid C in \mathbf{R}^3 given in spherical coordinates by $R = [0, a]$, $\theta = [0, 2\pi]$, $\varphi = [0, \pi/4]$ has a density function $\rho(R, \theta, \varphi) = 4R \sin \frac{\theta}{2} \cos \varphi$. Find the total mass of C .

(10)

4. (a) A variable plane is at a constant distance p from the origin and meets the axes in A , B and C . Show that the locus of the centroid of the tetrahedron $OABC$ is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}.$$

(10)

- (b) Find the locus of the point of intersection of perpendicular generators of a hyperboloid of one sheet.

(10)

- (c) Planes are drawn through a fixed point (α, β, γ) so that their sections of the paraboloid $ax^2 + by^2 = 2z$ are rectangular hyperbolas. Prove that they touch the cone

$$\frac{(x-\alpha)^2}{b} + \frac{(y-\beta)^2}{a} + \frac{(z-\gamma)^2}{a+b} = 0.$$

(10)

- (d) Show that the enveloping cylinder of the conicoid

$$ax^2 + by^2 + cz^2 = 1$$

with generators perpendicular to z -axis meets the plane $z = 0$ in parabolas.

(10)

SECTION B

5. Answer any four of the following

- (a) Form the differential equation that represents all parabolas each of which has latus rectum $4a$ and whose axes are parallel to the x -axis.

(10)

- (b) (i) The auxiliary polynomial of a certain homogeneous linear differential equation with constant coefficients in factored form is $P(m) = m^4(m-2)^6(m^2-6m+25)^3$. What is the order of the differential equation and write a general solution?

(5)

- (ii) Find the equation of the one-parameter family of parabolas given by $y^2 = 2cx + c^2$, c real and show that this family is self-orthogonal.

(5)

- (c) A circular wire of radius a and density ρ attracts a particle according to

$$\gamma \frac{m_1 m_2}{(\text{distance})^2}.$$

If the particle is placed on the axis of the wire at a distance b from the centre find its velocity when it is at a distance x . If it is placed at a small distance from the centre on the axis show that the time of a complete oscillation is

$$a \sqrt{\frac{2\pi}{\gamma \rho}}$$

(10)

- (d) A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freely jointed together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table. If C and F be connected by a light string, prove that the tension in the string is $W\sqrt{3}$.

(10)

- (e) For the curve

$$\vec{r} = a(3t-t^3)\vec{i} + 3at^2\vec{j} + a(3t+t^3)\vec{k}, \quad a$$

being a constant. Show that the radius of curvature is equal to its radius of torsion.

(10)

6. (a) Solve and examine for singular solution the following equation.

$$p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0.$$

(10)

- (b) Solve the differential equation

$$\frac{d^2 y}{dx^2} + 9y = \sec 3x,$$

(10)

- (c) Given $y = x + \frac{1}{x}$ is one solution, solve the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0$$

by reduction of order method.

- (d) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2y \frac{dy}{dx} - 3y = 2e^x - 10 \sin x$$

by the method of undetermined coefficients.

(10)

7. (a) If two particles are projected in the same vertical plane with velocities u and u' at angles α and α' with the horizontal, show that the interval between their transits through the other point common to their paths is $\frac{2 u u' \sin(\alpha - \alpha')}{g u \cos \alpha + u' \cos \alpha'}$ assuming $\alpha > \alpha'$.

(10)

- (b) Two particles of masses m and M move under the force of their mutual attraction, if the orbit of m relative to M is a circle of radius a described with velocity v , show that

$$v = \left[\frac{G(M+m)}{a} \right]^{1/2}$$

(10)

- (c) Show that the length of an endless chain which will hang over a circular pulley of radius r so as to be in contact with two-thirds of the circumference of the pulley is

$$r \left\{ \frac{3}{\log(2 + \sqrt{3})} + \frac{4\pi}{3} \right\}$$

(10)

- (d) Prove that a circular cylinder of radius a and length $\frac{a}{n}$ cannot float upright in water in stable equilibrium if its specific gravity lies between

$$\frac{1}{2} \left[1 - \sqrt{1 - 2n^2} \right] \text{ and } \frac{1}{2} \left[1 + \sqrt{1 - 2n^2} \right].$$

What will happen if $2n^2 > 1$?

(10)

8. (a) Find $f(r)$ if $f(r) \vec{r}$ is both solenoidal and irrotational.

(10)

- (b) Evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant.

(10)

- (c) Verify the divergence theorem for $\vec{F} = 4x\vec{i} - 2y\vec{j} + z^2\vec{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

(10)

- (d) By using vector methods, find an equation for the tangent plane to the surface $z = x^2 + y^2$ at the point $(1, -1, 2)$.

(10)

MATHEMATICS

PAPER -II SECTION A

1. Answer any four parts

(10 × 4 = 40)

- (a) Show that the set of cube roots of unity is a finite Abelian group with respect to multiplication.
- (b) Evaluate the double integral $\iint_R x^2 dx dy$ where R is the region bounded by the line $y = x$ and the curve $y = x^2$.
- (c) Show that the function f defined by

$$f(x) = \frac{1}{x}, x \in [1, \infty)$$
 is uniformly continuous on $[1, \infty)$.
- (d) If f analytic, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2$$
- (e) Find the basic feasible solutions of the following system of equations in a linear programming problem

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ 2x_1 + x_2 + 5x_3 &= 5 \\ x_j &\geq 0, j = 1, 2, 3 \end{aligned}$$

2. (a) Show that the set $S = \{1, 2, 3, 4\}$ forms an Abelian group for the operation of multiplication modulo 5.

(14)

(b) Show that

(i) $h(x) = \sqrt{x + \sqrt{x}}, x \geq 0$ is continuous on $[0, \infty)$;

(ii) $h(x) = e^{\sin x}$ is continuous on \mathbb{R} .

(13)

(c) If $f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2}$, when $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, show that at $(0, 0)$

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$$

(13)

3. (a) Prove that the set of all real numbers of the form $a + b\sqrt{2}$, where a and b are real numbers, is a field under the usual addition and multiplication.

(13)

(b) Show that the transformation

$$w = \frac{5 - 4z}{4z - 2}$$

maps unit circle $|z| = 1$ onto a circle of radius unity and centre at $-\frac{1}{2}$.

(13)

- (c) Use contour integration technique to find the value of

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta}$$

(14)

4. (a) If R is commutative ring with unit element and M is an ideal in R , then show that M is maximal ideal if R/M is a field.

(13)

- (b) Solve the linear programming problem

Find mm $(8x_1 + 6x_2)$

subject to the constraints

$$4x_1 + 3x_2 \geq 18$$

$$2x_1 + 5x_2 \geq 16$$

$$x_1, x_2 \geq 0$$

using graphical method. Show that more than one feasible solution will yield the minimum of the objective function. Interpret this fact geometrically.

(13)

- (c) Use simplex method to solve the following linear programming problem.

(14)

Maximize $Z = 2x_1 - x_2 + 3x_3$

subject to the constraints

$$3x_1 + x_2 - 2x_3 \leq 6$$

$$2x_1 + 5x_2 + x_3 \leq 14$$

$$x_1 + 4x_2 + 2x_3 \leq 8$$

$$x_1, x_2, x_3 \geq 0$$

- (b) Show that the transformation

SECTION B

5. Answer any four parts

(10×4 = 40)

- (a) Apply Charpit's method to solve the equation

$$2z + p^2 + qy + 2y^2 = 0$$

- (b) Perform four iterations of the bisection method to obtain a positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0$$

- (c) Evaluate $\int_0^1 \sqrt{1+2x} dx$ by applying Gaussian quadrature formula, namely

$$\int_{-1}^1 f(t) dt = \sum_{i=1}^n A_i f(t_i)$$

where the coefficients A_i and the roots t_i are given below for $n = 4$ as

$$\begin{array}{ll} t_1 = -0.8611 & A_1 = A_4 = 0.3478 \\ t_2 = -0.3399 & A_2 = A_3 = 0.6521 \\ t_3 = 0.3399 & \\ t_4 = 0.8611 & \end{array}$$

(d) Evaluate the following expressions:

- (i) 78 OR 87
- (ii) 78 XOR 87
- (iii) 78 AND 87
- (iv) Shift 87 left by 2
- (v) Rotate 78 right by 2

(e) Find the moment of inertia of a uniform triangular lamina about one side.

6. (a) Solve $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ given that

(i) $u = 0$, when $x = 0$ for all t

(ii) $u = 0$, when $x = l$ for all t

$$\left. \begin{aligned} \text{(iii)} \quad u &= \frac{bx}{a}, \quad 0 < x < a \\ &= \frac{b(l-x)}{l-a}, \quad a < x < l \end{aligned} \right\} \text{at } t = 0$$

(iv) $\frac{\partial u}{\partial t} = 0$ at $t = 0$, x in $(0, l)$

(13)

(b) Solve:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = y \cos x$$

(13)

(c) A solid circular cylinder of radius a rotating about its axis is placed gently with its axis horizontal on a rough plane, whose inclination to the horizon is α . Initially, the friction acts up the plane and the coefficient of friction is μ . Show that the cylinder will move upwards, if $\mu > \tan \alpha$. Also, show that the time that lapses before rolling commences, is

$$\frac{a\Omega}{g(3\mu \cos \alpha - \sin \alpha)}$$

where Ω is the initial angular velocity of the cylinder.

(14)

7. (a) Apply Gauss-Seidel iterative method for five iterations to solve the equations :

(14)

$$\begin{bmatrix} 10 & -2 & -1 & -1 \\ -2 & 10 & -1 & -1 \\ -1 & -1 & 10 & -2 \\ -1 & -1 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ 27 \\ -9 \end{bmatrix}$$

(b) A two-dimensional flow field is given by $\psi = xy$. Then

- (i) show that the flow is irrotational;
- (ii) find the velocity potential;
- (iii) verify that P and its complex conjugate \bar{p} satisfy the Laplace equation;
- (iv) find the streamlines and potential lines.

(13)

(c) Write a BASIC program to evaluate a definite integral

$$\int_0^1 (x^3 + \sin x) dx$$

by Simpson's one-third rule. Indicate the lines which are to be modified for a different problem.

(13)

8. (a) Write a program in BASIC to solve the equation

$$x^3 - 4x^2 + x + 6 = 0$$

by Newton-Raphson method by taking the initial approximation as $x_0 = 5$. Indicate which lines are to be changed for a different equation.

(13)

- (b) Apply Runge-Kutta method of fourth order to find an approximate value of y when $x = 0.2$, given that

$$\frac{dy}{dx} = x + y^2, y = 1 \text{ when } x = 0$$

(14)

- (c) Determine the restrictions on f_1 , f_2 and f_3 , if

$$\frac{x^2}{a^2} f_1(t) + \frac{y^2}{b^2} f_2(t) + \frac{z^2}{c^2} f_3(t) = 1$$

is a possible boundary surface of a liquid.

(13)