MATHEMATICS

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PAPER - I

SECTION A

1. Answer any four of the following:

(a) Let U and W be subspaces of R^3 for which dim U = 1, dim W = 2 and $U \not\subset W$. Show that $R^3 = U + W$ and $U \cap W = \{0\}$.

(10)

(b) Let $\{e_1, e_2, e_3\}$ be the standard basis of R^3 and T be a linear transformation from R^3 into R^2 defined by $T(e_1) = (2, 3)'$, $T(e_2) = (1, 2)'$ and $T(e_3) = (-1, -4)'$(where "'" means transpose).

(10)

(i) What is T (1, -2, -1)?

(ii) What is the matrix of T with respect to the standard bases of R^3 and R^2?

(10)

(c) Using Lagrange's mean value theorem, show that

$$1 - x < e^{-x} < 1 - x + \frac{x^2}{2}, x > 0$$

(10)

(d) $$f(x,y) = \begin{cases} \frac{xy(x^2 - y^2)}{(x^2 + y^2)}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

(10)

(e) Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 2x - 4y + 5 = 0$, $x - 2y + 3z + 1 = 0$ is a great circle.

(10)

2. (a) Find a linear map $T: R^3 \rightarrow R^4$ whose range is generated by (1, 2, 0, -4) and (2, 0, -1, -3). Also, find a basis and the dimension of the

(i) range U of T.

(ii) kernel W of T.

(10)

(b) Find the eigen values and their corresponding eigen vectors of the matrix $\begin{pmatrix} 2 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. Is the matrix diagonalizable?

(10)

(c) For what values of a has the system of equations

$$x + 2y + z = 1,$$
$$\alpha x + 4y + 2z = 2,$$
$$4x - 2y + 2\alpha z = -1$$

(i) a unique solution

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(ii) infinitely many solutions

(iii) no solution ?

(10)

(d) Determine an orthogonal matrix which reduces the quadratic form

$$\left|Q(x_1,x_2,x_3) = 2x_1^2 + x_2^2 - 4x_2x_3 + x_3^2\right|$$

to a canonical form. Also, identify the surface represented by $Q(x_1, x_2, x_3) = 7$.

(10)

3. (a) Using Lagrange's multipliers, find the volume of the greatest rectangular parallelopiped that can be inscribed in the sphere $x^2 + y^2 + z^2 = 1$.

(10)

(b) Evaluate the integral $\iint_R \frac{xe^{-y^2}}{y}\,dxdy$, where R is the triangular region in the first quadrant bounded by $y = x$ and $x = 0$.

(10)

(c) Evaluate $\int_0^1 x^m \left(\ln\frac{1}{x}\right)^n dx, m, n > -1$.

(10)

(d) Find the volume cutt off the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$

(10)

4. (a) A variable plane is at a constant distance p from the origin and meets the axes at A, B and C. Through A, B and C, the planes are drawn parallel to the coordinate planes. Find the locus of their point of intersection.

(10)

(b) Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1,-2,1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

(10)

(c) Find the equation of the right circular cone generated by straight lines drawn from the origin to cut the circle through the three points (1, 2, 2), (2, 1, -2) and (2, -2, 1).

(10)

(d) Find the equations of the tangent planes to the ellipsoid $7x^2 + 5y^2 + 3z^2 = 60$ which pass through the line $7x + 10y - 30 = 0, 5y - 3z = 0$.

(10)

SECTION B

5. Answer any four of the following:

(a) Determine the family of orthogonal trajectories of the family $y = x + ce^{-x}$.

(10)

(b) Show that the solution curve satisfying $(x^2 - xy)\, y' = y^3$, where $y \to 1$ as $x \to 1, x \to \infty$ is a conic section. Identify the curve.

(10)

(c) A particle moves with a central acceleration which varies inversely as the cube of the distance; if it be projected from an apse at a distance a from the origin with a velocity $\sqrt{2}$ times the velocity for a circle of radius a, determine the path.

(10)

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(d) A heavy uniform chain AB hangs freely under gravity with A fixed and B attached by a light string BC to a fixed point C at the same level as A. The chain AB and the string BC make angles 60^0 and 30^0 respectively with the horizontal. Find the ratio of the length of the string to that of the chain.

(10)

(e) Evaluate $\int_c \vec{F}.\vec{dr}$ for the field $\vec{F}$ = grad (xy^2z^3) where c is the ellipse in which the plane $z = 2x + 3y$ cuts the cylinder $x^2 + y^2 = 12$ counterclockwise as viewed from the positive end of the z-axis looking towards the origin.

(10)

6. (a) Solve $(1+x)^2 y''+(1+x)y'+y=4\cos(\ln(1+x)), y(0)=1, y(e-1)=\cos 1$

(10)

(b) Obtain the general solution of $y''+2y'+2y=4e^{-x}x^2\sin x$

(10)

(c) Find the general solution of $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)dy = 0$.

(10)

(d) Obtain the general solution of $(D^4 + 2D^3 - 2D)\, y = x + e^{2x}$, where $Dy = \frac{dy}{dx}$

(10)

7. (a) A particle is projected along the inside of a smooth vertical circle of radius 18 cm from the lowest point. Find the velocity of projection so that after leaving the circle, the particle may pass through the centre.

(14)

(b) Three forces each of magnitude P and acting in the positive directions of the axes have their lines of action

$$-y = z = 2, -z = x = 2, -x = y = 2.$$

Show that they are equivalent to a force at the origin and a couple. Determine the magnitude of the force and the moment of the couple.

(12)

(c) A circular cone, whose vertical angle is 60° has its lowest generator horizontal and is filled with liquid. Prove that the resultant pressure on the curved surface is $\frac{\sqrt{19}}{2}$ times the weight of the liquid.

(14)

8. (a) Show that

$div\left(\vec{A}\times\vec{B}\right) = \vec{B}.curl\,\vec{A} - \vec{A}.curl\vec{B}$ (10)

(b) Evaluate curl $\left[\frac{(2\vec{i}-\vec{j}+3\vec{k})\times\vec{r}}{r^n}\right]$ Where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r^2 = x^2 + y^2 + z^2$

(10)

(c) Evaluate $\iint_S \left(x\vec{i} + y\vec{j} + z\vec{k}\right).\vec{n}dS$, where S is the surface $x + y + z = 1$ lying in the first octant.

(10)

(d) Evaluate $\nabla^2 u$ in spherical polar coordinates.

(10)

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MATHEMATICS

PAPER - II

SECTION A

1. Answer any four parts:

(10 x 4 = 40)

(a) If every element, except the identity, of a group is of order 2, prove that the group is abelian.

(b) Show that the sequence (f_n), where

$$f_n(x) = nxe^{-nx^2}$$

is pointwise, but not uniformly convergent in $[0, \infty)$.

(c) Investigate the continuity at (0, 0) of the function

$$f(x, y) = \begin{cases} \dfrac{(x^2 - y^2)}{(x^2 + y^2)}, & (x, y) \neq (0,0) \\ 0 & ,(x, y) = (0,0). \end{cases}$$

(d) Find the analytic function $f(z) = u(x, y) + iv(x, y)$ for which $u - v = e^x(\cos y - \sin y)$.

(e) Prove that $x_1 = 2$, $x_2 = 3$, $x_3 = 2$ is a feasible solution, but not a basic feasible solution, to the set of constraints.

$$x_1 + x_2 + 2x_3 = 9$$
$$3x_1 + 2x_2 + 5x_3 = 20, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Also find all basic feasible solutions of the system.

2. (a) Prove that the set $R = \{a + \sqrt{2}b, a, b \in I\}$ is a ring. Is it an integral domain? Justify your answer.

(13)

(b) Evaluate $\int_{-1}^{1} f(x)dx$, where $f(x) = |x|$, by Riemann integration.

(14)

(c) Find the bilinear transformation maps $z = 1, 0, \infty$ to $w = 0, \infty, 1$ respectively.

(13)

3. (a) Show that $f(x, y) = x^4 + x^2y + y^2$ has a minimum at (0, 0).

(13)

(b) Find the singular points with their nature and the residues thereat of

$$f(x) = \frac{\cot \pi z}{\left(z - \frac{1}{3}\right)^2}$$

(13)

(c) A company has three factories F_1, F_2, F_3 and three warehouses W_1, W_2, W_3. The supplies are transported from the factories to the warehouses. The cost in rupees for transportation of the product from the factories to the warehouses are shown below:

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	W_1	W_2	W_3	Factory capacity in units
F_1	8	10	12	900
F_2	12	13	12	1000
F_3	14	10	11	1200
Warehouse requirement in units	1200	1000	9000	

Assign factory capacities to warehouse requirements to minimize the cost of transportation.

(14)

4. (a) Prove that a function, analytic for all finite values of z and bounded, is a constant.

(13)

(b) Let G be a group of real numbers under addition and G' be a group of positive real numbers under multiplication. Show that the mapping $f: G \to G'$ defined by $f(a) = 2^a \; \forall a \in G$ is a homomorphism. Is it an isomorphism too? Supply reasons.

(13)

(c) Using Simplex algorithm solve the LPP

Min $z = x_1 - 3x_2 + 2x_3$

subject to

$$3x_1 - x_2 + 2x_3 \le 7$$

$$-2x_1 + 4x_2 \le 12$$

$$-4x_1 + 3x_2 + 8x_3 \le 10$$

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$$

(14)

SECTION B

5. Answer any four parts

(10 x 4 = 40)

(a) Find the general solution of the partial differential equation

$(z^2 - 3yz - y^2)\,p + x(y + z)\,q = x(y - z)$

(b) If the Lagrangian L of a dynamical system does not involve t explicitly, prove that the Hamiltonian H of the system is constant and is equal to the total energy.

(c) If w is the area of cross-section of a stream filament, prove that the equation of continuity is

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial s}(\rho w q) = 0,$$

where δs is an element of arc of the filament in the direction of flow and q is the speed.

(d) Using Newton-Raphson method obtain a root near $x = 0$, and correct to three decimal places of the equation $x + \sin x = 1$.

(e) Convert

(i) the decimal number 412 to octal, to binary and finally to hexadecimal number.

(ii) the hexadecimal number F9A. BC3 to a decimal number.

6. (a) Apply Charpit's method to find the complete integral of the partial differential equation $pxy + pq + qy = yz$.

(13)

(b) A uniform rod AB is held in a vertical position with the end A resting on a perfectly rough table. When the rod is released, it rotates about the end in contact with the table. Prove that the end A of the rod does not leave the table.

(14)

(c) Write a BASIC program to evaluate

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$$\int_{2}^{5} \frac{dx}{1+x^2}$$

using Simpson's one-third rule with 20 subintervals.

(13)

7. (a) Solve the initial value problem

$$\frac{dy}{dx} = \frac{1}{x+y}, \; y(0) = 1$$

using Runge-Kutta method of fourth order to evaluate y(0, 5) in a single step.

(13)

(b) A sphere of radius a is surrounded by infinite liquid of density ρ, the pressure at infinity being Π. The sphere is suddenly annihilated. Show that the pressure at a distance r from the centre of the sphere immediately falls to $\Pi\left(1-\frac{a}{r}\right)$.

(13)

(c) Solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; 0 \le x \le \pi, \quad 0 \le y < \infty,$$

$u(0, y) = 0$, for $0 \le y < \infty$,
$u(\pi, y) = 0$, for $0 \le y < \infty$,
$u(x, \infty) = 0$, for $0 \le x \le \pi$,
and $u(x, 0) = u_0$, for $0 \le x \le \pi$,

(14)

8. (a) Using Gauss-Seidel iteration method find the solution, correct to three decimal places, of the linear system

$7x + 52y + 13z = 104$
$3x + 8y + 29z = 71$
$83x + 11y - 4z = 95$

with $(x^0, y^0, z^0) = (1.145, 1.846, 1.821)$. Only two iterations may be supplied.

(13)

(b) Find the moment of inertia of an elliptic area of mass M and semi-axes a and b about a diameter of length 2r.

(13)

(c) Prove that the image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.

(14)

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