MATHEMATICS

PAPER-I

SECTION A

1. Attempt any four of the following:

(i) Let C (R) be the vector space of complex numbers over the field of real numbers. Under what conditions on the real numbers α, β, γ, δ in the set

$S = \{\alpha + i\beta, \gamma + i\delta\}\ i = (-1)^{1/2}$

do we have

L(S) = Cm),

where L(S) denotes the linear span of S. Justify your claim. (10)

(ii) Sketch the conic

$$[x \quad y \quad 1]\begin{bmatrix} 5 & 3 & -5 \\ 3 & 5 & -3 \\ -5 & -3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

(10)

(iii) A plane passes through a fixed point (2p, 2q, 2r) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is

$$\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 1$$

(10)

(iv) Prove that the stationary values of

$$u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$$

subject to the conditions, $lx + my + nz = 0$ and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

are the roots of the equations

$$\frac{l^2a^4}{1-a^2u} + \frac{m^2b^4}{1-b^2u} + \frac{n^2c^4}{1-c^2u} = 0$$

(10)

(v) Evaluate

$$\int_{-1}^{1} x^3$$

dx from first principles by using

Riemann's theory of integration. (10)

2. (a) Let

$D : V(R) \to V(R) ; f(x) \to \dfrac{df(x)}{dx}$,

$T : V(R) \to V(R) ; f(x) \to xf(x)$,

be linear transformations on V(R), the vector space of all polynomials in an indeterminate x with real coefficients.

Show that:

(i) DT — TD = I, the identity operator.
(ii) $(TD)^2 = T^2D^2 + TD$.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

by using Cayley-Hamilton theorem. (20)

(b) Show that every square matrix can be expressed uniquely as A + iB where A, B are Hermitian.

Reduce the quadratic form

$x^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6zx - 4xy - 2xt - 6zt$ to canonical form and find its rank and signature.

(20)

3. (a) If $f(x)$ is a continuous function of x satisfying $f(x+y) = f(x)+f(y)$, for all real numbers x, y, then prove that $f(x) = Ax$, for all real numbers x, where A is a constant.

Express

$$y = \left(x + \sqrt{1+x^2}\right)$$

in ascending powers of x, by Taylor's theorem. (20)

(b) Using the transformations

$$u = \frac{x^2+y^2}{x},\ v = \frac{x^2+y^2}{y}$$

Evaluate

$$\iint \frac{(x^2+y^2)^2}{x^2y^2}\, dx\, dy$$

over the area common to the circles

$x^2 + y^2 - ax = 0$ and $x^2 + y^2 - by = 0$

Evaluate

$$\iiint (1-x-y-z)^{l-1} x^{m-1} y^{n-1} z^{p-1}\, dx\, dy\, dz$$

over the interior of the tetrahedron bounded by the planes

$x = 0, y = 0, z = 0, x + y + z = 1.$

(20)

4. (a) A straight line AB of fixed length moves so that its extremities A, B lie on two fixed straight lines OP, OQ inclined to each other at an angle to. Prove that the locus of the circum centre of ΔOAB is a circle.

Find the locus of the lines which move parallel to the zx-plane and meet the curves

$xy = c^2, z = 0$

$y2 = 4cz, x = 0$

(20)

(b) Two cones with a common vertex pass through the curves

$y = 0, z^2 = 4ax$

$x = 0, z^2 = 4by$

The plane z = 0 meets them in two conics which intersect in four concyclic points. Show that vertex lies on the surface

$$z^2\left(\frac{x}{a}+\frac{y}{b}\right)=4\left(x^2+y^2\right)$$

Find the radii of curvature and torsion at any point of the curve

$x^2 + y^2 = a^2$, $x^2 - y^2 = az$. (20)

SECTION B

5. Attempt any four of the following:

(i) An amount of x rupees is invested at an interest of k% per annum compounded continuously. If the differential equation satisfying the above is given be

$$\frac{dx}{dt}=(0.01)\,kx$$

and if an amount of Rs. 100 is invested at 15% per annum compounded continuously, find the amount at the end of 5 years.

(10)

(ii) Solve for x the vector equation

$$px + x \times a = b,\ p \neq 0$$

(10)

(iii) Suppose we are given that A_i^j and C_p^q are tensors related through

$$A_q^{\,j}\,B(k,p)=C_q^{\,j}.$$

Prove that B (k,p) is a tensor of contravariant rank 1 and convariant rank 1.

(10)

(iv) Let u be the speed of a particle and c the speed of light. For what value of u/c will the relativistic mass of a particle exceed its rest mass by a given fraction *f*?

(10)

(v) A number of particles are projected simultaneously from the same point with equal velocities in the same vertical plane in different directions. Find the locus of the foci of their paths.

(10)

6. (a) Solve

$$(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0,\ p=\frac{dy}{dx}$$

Interpret geometrically the factors in the p-and c-discriminants of the equation

$$8p^3x = y\,(12p^2-9)$$

(20)

(b) Solve:

$$(i)\ \frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}-\frac{a^2}{x^4}y=0.$$

$$(ii)\ \frac{d^2y}{dx^2}+(\tan x-3\cos x)\frac{dy}{dx}+2y\cos^2 x=\cos^4 x$$

by varying the parameters. (20)

7. (a) Use the principle of virtual work, to find the inclination of a rod to the horizontal, when the rod lies in equilibrium with its ends on two smooth planes inclined at angles 30^0 and 60° to the horizontal, the planes intersecting in a horizontal line.

A hollow gas-tight balloon containing helium, weighs W kilograms when its lowest point touches the ground it requires a force of w kilograms to prevent it from rising. Show that it can float in equilibrium at a height

$$H \log_e \left(1 + \frac{w}{W}\right)$$

where H is a certain constant.

(20)

(b) Show that the electromagnetic wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0$$

retains its form under Lorentz transformation but not under Galilean transformation.

At what rate is the sun losing rest mass due to its radiation, given that

(i) the earth receives radiant energy from the sun at the rate of 1.34×10^{11} watts/metre^2.

(ii) the sun radiates isotropically.

(iii) the mean earth-sun separation = 1.49×10^{11} metres;

(iv) $c^2 = 8.99 \times 10^{16}$ joules/kg.

(20)

8. (a) Prove the identities

(i) Curl grad $\phi = 0$.

(ii) Div curl f=0.

If $\overrightarrow{OA} = a\mathbf{i}$, $\overrightarrow{OB} = a\mathbf{j}$, $\overrightarrow{OC} = a\mathbf{k}$ form three coterminous edges of a cube and S denotes the surface of the cube, evaluate

$$\int_S \left\{(x^3 - yz)\,\mathbf{i} - 2x^2y\,\mathbf{j} + 2\mathbf{k}\right\} \cdot \mathbf{n}\, dS$$

by expressing it as volume integral, where n is the unit outward normal to dS.

(b) Prove that the double inner product $A^{ab} S_{ab}$ of the skew symmetric tensor A^{ab} and the symmetric tensor S_{ab} vanishes identically

Verify that

$$\frac{\partial A^i}{\partial x^i}$$

is not a tensor, but

$$\frac{\partial A^i}{\partial x^j} + a^k \Gamma^i_{kj}$$

is a tensor of contravariant rank one and covariant rank one, by assuming the law of transformation of the Christoffel symbols Γ^i_{kj}

(20)

MATHEMATICS

PAPER-II

SECTION A

1. Answer any four parts

(a) Show that an infinite cyclic group is isomorphic to the additive group of integers.

(b) Change the order of integration in

$$\int_0^{2a}\int_{\frac{x^2}{4a}}^{3a-x}\phi(x, y)\,dx\,dy$$

(c) Expand the function f(z) = log (z + 2) in a power series and determine its radius of convergence.

(d) By applying Newton-Raphson method to

$$f(x) = 1 - \frac{a}{x^n},$$

prove that

$$x_{k+1} = \frac{1}{n}\left[(n+1)\,x_k - \frac{x_k^{n+1}}{a}\right]$$

(e) Test the convergence of the integral

$$\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$$

2. (a) Show that every finite integral domain is a field.

(b) Show that every finite field is a field extension of field of residues modulo a prime p.

(c) Test for uniform convergence the series

$$\sum_{n=1}^{\infty} 2^n \frac{x(2^n-1)}{1+x^{2n}}$$

3. (a) Prove that the function

$$f(x, y) = x^2 - 2xy + y^2 - x^3 - y^3 + x^5$$

has neither a maximum nor a minimum at the origin.

(b) Evaluate

$$\int_0^{\infty} \frac{\log_e(x^2+1)}{x^2+1}\,dx$$

by using method of residues.

(a) Prove that the function f(z) = a + iv, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$f(0) = 0$$

satisfies Cauchy-Riemann equations at the origin, but f' (0) does not exist.

4. (a) Define interpolation. Find the polynomial $P_2(x)$ which satisfies

$$f(-1) = P_2(-1) = 2$$
$$f(1) = P_2(1) = 1$$
$$f(2) = P_2(2) = 1$$

Find $f(1.5)$.

(b) Discuss Simpson's one-third rule of integration.

Use it to find the value of

$$\int x \, dx.$$

(c) Draw a flow chart for finding the roots of the equation $ax^2 + bx + c = 0$.

SECTION B

5. Answer any four parts:

(a) Solve the following initial value problem

$$(y+z)z_x + yz_y = x - y;$$
$z = 1 + t$ on the initial curve C :
$$x = t, y = 1; -\infty < T < \infty$$

(b) A Lagrangian for a particular physical system can be written as

$$L = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2)$$

$$\frac{K}{2}(ax^2 + 2bxy + cy^2)$$

where a, b and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. Obtain the equations of motion.

(c) Let X and Y be two random variables taking three values-1, 0, 1 having the following joint probability distribution

Y \ X	-1	0	1	Total
-1	0	0.1	0.1	0.2
0	0.2	0.2	0.2	0.6
1	0	0.1	0.1	0.2
Total	0.2	0.4	0.4	

(i) Fine E(X), E(Y) and E(XY). Are X and Y independent?

(ii) Derive conditional probability distribution of X given that Y = 0.

(d) Two companies A and B are competing for the same product. Their different strategies are given in the following pay off matrix:

		Company B	
		B_1	B_2
Company A	A_1	5	1
	A_2	3	4

Using linear programming or otherwise determine the best strategies for the company B.

(e) Give the arrangement of sources and sinks that will give rise to the function (complex potential)

$$W = \log\left(z - \frac{a^2}{z}\right)$$

Draw a rough sketch of stream lines and prove that two of them subdivide into the circle r = a and the axis of y.

6. (a) Determine the complete integral of the equation

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial x}\right)^2\left(x\frac{\partial z}{\partial y} + \left[\frac{\partial z}{\partial x}\right]^2\right) + \left(\frac{\partial z}{\partial y}\right)^2$$
$$\left(y\frac{\partial z}{\partial x} + \left[\frac{\partial z}{\partial y}\right]^2\right)$$

(b) Solve the following partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\,\partial y} - 2\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} = 0$$

(c) For a certain mass pulley system the kinetic energy T is given by

$$T = \tfrac{1}{2}m_1(\dot y + \dot y)^2 + m_2(\dot y - \dot y_1)^2 + \tfrac{1}{2}M\dot y^2 + \tfrac{1}{2}\frac{I}{R^2}\dot y_1^2.$$

Write down the Lagrange's equation of motion corresponding to y and y_1 and show that

$$\ddot y = -k\left(\frac{B}{AB - C^2}\right)(y - y_0) + g$$
$$B\ddot y_1 - C\ddot y = Cg$$

Where

$$A = m_1 + m_2 + M,\ B = m_1 + m_2 + \frac{I}{R^2},\ C = m_1 - m_2$$

and y_0 is the value of y when the spring is unstretched.

7. (a) Find the moment of inertia of a solid circular cone of height h and vertical angle 2α, about a line through the vertex parallel to the base.

(b) The region $0 \le z \le h$ between the planes z = 0, z = h is filled with incompressible viscous fluid. The plane z = 0 is held at rest and the plane z = h moves with constant velocity V_j. It is required to determine the nature of the flow when conditions are steady, assuming there is no slip between the fluid and either boundary, neglecting body forces.

(c) If x = 4y + 5 and y = kx + 4 are the regression lines of x on y and of y on x respectively, then show that $0 \le 4k \le 1$.

If $k = \frac{1}{16}$, find the means of x and y and the coefficient of correlation between them.

8. (a) Let $\{X_k\}$ be a sequence of independent and identically distributed randomvariables such that $EX_k = 0$ and $\sigma_{X_k} = \sigma < \infty$, then show that as

$$n \to \infty,\ \sqrt{n}\left(\bar X_n\right)/\sigma \to N(0, 1)$$

in probability.

(b) A toy company manufactures two types of doll doll type A and doll type B. Each doll of type B takes twice as long to produce as one of type A and the company could have time to make a maximum of 800 dolls of type A per day. The supply of plastic is sufficient to produce 600 dolls per day of both types. The type B doll requires a fancy dress of which there are only 200 available per day. If the company makes a profit of Rs. 20 and Rs. 30 per doll on type A and B respectively, then how many of each doll should be produced per day in order to maximise the total profit?

(c) A company has 3 warehouses and 3 stores, the cost of shipping one unit from warehouse i to store j is given in the following table:

		To store j		
		1	2	3
From warehouse i	1	0	2	0
	2	1	4	0
	3	0	2	4

If the requirements of the three stores are 70, 50, 30 respectively and the quantities available at the warehouses are 70, 30, 50 respectively, then find the minimum cost solution.