C.S.E-Mains 2008

MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

PAPER – I

SECTION-A

1. Attempt any FIVE of the following:

(a) Show that the matrix A is invertible if and only if the adj (A) is invertible. Hence find |adj (A)|.

(12)

(b) Let S be a non-empty set and let V denote the set of all functions from S into R. Show that V is a vector space with respect to the vector addition (f + g)(x) = f(x) + g(x) and scalar multiplication (c.f) (x) = cf (x).

(12)

(c) Find the value of $\lim_{x\to 1}(1-x)\cot\frac{\pi x}{2}$

(12)

(d) Evaluate $\int_0^1 (x\,\ell n x)^3\,dx$

(12)

(e) The plane x–2y+3z = 0 is rotated through a right angle about its line of intersection with the plane 2x + 3y –4z –5 = 0; find the equation of the plane in its new position.

(12)

(f) Find the equations (in symmetric form) of the tangent line to the sphere $x^2 + y^2 + z^2 + 5x - 7y + 2z - 8 = 0$, $3x - 2y + 4z + 3 = 0$ at the point (–3, 5, 4).

(12)

2. (a) Show that B = {(1, 0, 0,) (1, 1, 0), (1, 1, 1)} is a basis of R^3. Let T : $R^3 \to R^3$ be a linear transformation such that T(1, 0, 0) = (1, 0, 0), T(1, 1, 0) = (1, 1, 1) and T(1, 1, 1) = (1, 1, 0). Find T(x, y, z).

(20)

(b) Determine the maximum and minimum distances of the origin from the curve given by the equation

$5x^2 + 4xy + 6y^2 = 140$.

(20)

(c) A sphere S has points (0, 1, 0), (3, –5, 2) at opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane 5x–2y + 4z + 7 = 0 as a great circle.

(20)

3. (a) Let a be a non-singular matrix.

Show that if

$I + A + A^2 + \ldots\ldots\ldots + A^n = 0$,

then $A^{-1} = A^n$.

(20)

(b) Evaluate the double integral

$$\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2 + y^2}$$

by changing the order of integration

(20)

(c) If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone

$$5yz - 8zx - 3xy = 0,$$

find the equations of the other two.

(20)

4. (a) Find the dimension of the subspace of R^4 spanned by the set

$\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$.

Hence find a basis for the subspace.

(20)

(b) Obtain the volume bounded by the elliptic paraboloide given by the equations

$$z = x^2 + 9y^2 \text{ and } z = 18 - x^2 - 9y^2.$$

(20)

(c) Show that the enveloping cylinders of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ with generators perpendicular to z axis meet the plane $z = 0$ in parabolas.

(20)

SECTION-B

5. Attempt any FIVE of the following:

(a) Solve the differential equation

$$ydx + (x + x^3y^2)dy = 0.$$

(12)

(b) Use the method of variation of parameters to find the general solution of

$$x^2y'' - 4xy' + 6y = -x^4 \sin x.$$

(12)

(c) A smooth parabolic tube is placed with vertex plane. A particle slides down the tube from rest under the influence of gravity. Prove that in any position, the reaction of the tube is equal to $2w\left(\frac{h+a}{\rho}\right)$, where 'w' is the weight of the particle, 'ρ' the radius of curvature of the tube, '4a' its latus rectum and 'h' the initial vertical height of the particle above the vertex of the tube.

(12)

(d) A straight uniform beam of length '2h' rests in limiting equilibrium, in contact with a rough vertical wall of height 'h', with one end on a rough horizontal plane and with the other end projecting beyond the wall. If both the wall and the plane be equally rough, prove that 'λ', the angle of friction, is given by $\sin 2\lambda = \sin\alpha \sin\alpha$, 'α' being the inclination of the beam to the horizon.

(12)

(e) Find the constant a and b so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.

(12)

(f) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential for $\vec{F}$ and the work done in moving an object in this field from $(1, -2, 1)$ to $(3, 1, 4)$.

(12)

6. (a) A particle P moves in a plane such that it is acted on by two constant velocities u and v respectively along the direction OX and along the direction perpendicular to OP, where O is

some fixed point, that is the origin. Show that the path trasversed by P is a conic section with focus at O and eccentricity u/v.

(15)

(b) Using Laplace transform, solve the initial value problem

$$y'' - 3y' + 2y = 4t + e^{3t}$$

with $y(0) = 1$, $y'(0) = -1$.

(15)

(c) Solve the differential equation

$$x^3 y'' - 3x^2 y' + xy = \sin(\ln x) + 1.$$

(15)

(d) Solve the equation

$$y - 2xp + yp^2 = 0, \text{ where } p = \frac{dy}{dx}.$$

(15)

7. (a) A particle of mass m moves under a force $m\mu\{3au^4 - 2(a^2 - b^2)u^5\}$, $u = \frac{1}{r} > b$, a, b, and $\mu(>0)$ being given constants. It is projected from an apse at a distance a + b with velocity $\frac{\sqrt{\mu}}{a+b}$. Show hat its orbit is given by the equation $r = a + b\cos\theta$, where (r, θ) are the plane polar coordinates of a point .

(15)

(b) A shell lying in a straight smooth horizontal tube suddenly breaks into two portions of masses m_1 and m_2. If s be the distance between the two masses inside the tube after time t, show that the work done by the explosion can be written as equal to

$$\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\cdot\frac{s^2}{t^2}$$

(15)

(c) A ladder of weight 10 kg rests on a smooth horizontal ground leaning against a smooth vertical wall at an inclination $\tan^{-1} 2$ with the horizon and is prevented from slipping by a string attached at its lower end and to the junction of the floor and the wall. A boy of weight 30 kg. begins to ascend the ladder. If the string can bear a tension of 10 kg. wt., how far along the ladder can the boy rise with safety?

(15)

(d) A solid right circular cone whose height is h and radius of whose base is r, is placed on an inclined plane. It is prevented from sliding. If the inclination of the plane θ (to the horizontal) be gradually decreased, find when the cone will topple over. For a cone whose semi-vertical angle is 30°, determine the critical value of θ which when exceeded, the cone will topple over.

(15)

8. (a) Prove that $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$ where $r = \left(x^2 + y^2 + z^2\right)^{1/2}$. Hence find f(r) such that $\nabla^2 f(r) = 0$.

(12)

(b) Show that for the space curve

$$x = t, y = t^2, z = \frac{2}{3}t^3,$$

the curvature and torsion are same at every point.

(15)

(c) Evaluate $\int_c \vec{A}.d\vec{r}$ along the curve $x^2 + y^2 = 1$, $z = 1$ from (0, 1, 1) to (1, 0, 1) if

$$\vec{A} = (yz + 2x)\hat{i} + xz\hat{j} + (xy + 2z)\hat{k}$$

(15)

(d) Evaluate $\iint_S \vec{F}.\hat{n}\, dS$ where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface of the cylinder bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

(15)

C.S.E-Mains 2008

MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

PAPER - II

SECTION A

1. Answer any FIVE of the following:

(5 x 12 = 60)

(a) Let R_0 be the set of all real numbers except zero. Define a binary operation * on R_0 as : a * b = | a | b where | a | denotes absolute value of a.
Does (R_0, *) form a group? Examine.

(12)

(b) Suppose that there is a positive even integer n such that $a^n = a$ for all the elements 'a' of some ring R. show that a + a = 0 for all a ∈ r and $a + b = 0 \Rightarrow a = b$ for all a, b ∈ R.

(12)

(c) (i) For x > 0, show $\frac{x}{1+x} < \log(1+x) < x$.

(6)

(ii) Let

$$T = \left\{\frac{1}{n}, n \in N\right\} \cup \left\{1 + \frac{3}{2n}, n \in N\right\} \cup \left\{6 - \frac{1}{3n}, n \in N\right\}.$$

Find derived set T' of T. Also find Supremum of T and greatest number of T.

(6)

(d) If f : R → R is continuous and
$f(x + y) = f(x) + f(y)$,
for all x, y ∈ □, then show that f(x) = x f(1)
for all x ∈ □

(12)

(e) Find the residue of
$$\frac{\cot z \coth z}{z^3} \text{ at } z = 0$$

(f) Find the dual of the following linear programming problem:
Max. $Z = 2x_1 - x_2 + x_3$
such that
$$x_1 + x_2 - 3x_3 \le 8$$
$$4x_1 - x_2 + x_3 = 2$$
$$2x_1 + 3x_2 - x_3 \ge 5$$
$$x_1, x_2, x_3 \ge 0$$

(12)

2. (a) Let G and $\bar{G}$ be two groups and let $\phi : G \to \bar{G}$ be a homomorphism. For any element a ∈ G
(i) prove that 0(ϕ(a))/0(a).

(ii) Ker ϕ is a normal subgroup of G.

(15)

(b) Let R be a ring with unity. If the product of any two non zero elements is non zero prove that $ab = 1 \Rightarrow ba = 1$.
Whether Z_6 has the above property or not explain. If Z_6 in integral domain?

(15)

(c) Discuss the convergence of the series

$$\frac{x}{2}+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\text{.......},x>0.$$

(15)

(d) Show that the series $\sum \frac{1}{n(n+1)}$ is equivalent to

$$\frac{1}{2}\prod_{2}^{\infty}\left(1+\frac{1}{n^2-1}\right).$$

(15)

3. (a) Prove that every Integral Domain can be embedded in a field.

(15)

(b) Show that any maximal ideal in the commutative ring F[x] of polynomials over a field f is the principal ideal generated by an irreducible polynomial.

(15)

(c) Let f be a continuous function on [0,1]. Using first mean value theorem on Integration, prove that

$$\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}dx=\frac{\pi}{2}f(0).$$

(15)

(d) (i) Prove that the sets
A = [0,1], B =]0, 1[are equivalent sets.

(6)

(ii) Prove that

$$\frac{\tan x}{x}>\frac{x}{\sin x}, x\in\left]0,\frac{\pi}{2}\right[.$$

(9)

4. (a) Evaluate

$$\int_C\left[\frac{e^z}{z^2(z^2+2z+2)}+\log(z-6)+\frac{1}{(z-4)^2}\right]dz$$

where C is the circle |z| = 3. State the theorems you use in evaluating above integral.

(15)

(b) Let f(z) be an entire function satisfying $|f(z)| \le k|z|^2$ for some positive constant k and all z. Show that $f(z) = az^2$ for some constant a.

(15)

(c) Solve the following transportation problem :

Destinations

		D_1	D_2	D_3	D_4	D_5	D_6	Availability
Factories	F_1	2	1	3	3	2	5	50
	F_2	3	2	2	4	3	4	40
	F_3	3	5	4	2	4	1	60
	F_4	4	2	2	1	2	2	30
	Demand	30	50	20	40	30	10	

by finding the initial solution by Matrix Minima method.

(30)

SECTION B

5. Answer any FIVE of the following:

(a) Find the general solution of the partial differentia. equation

$$(2xy-1)p+(z-2x^2)q=2(x-yz)$$

and also find the particular solution which passes through the lines x = 1, y = 0.

(12)

(b) Find general solution of the particle differential equation :

$$(D^2+DD'-6D'^2)z=y\cos x,$$

where $D=\frac{\partial}{\partial x}, D'=\frac{\partial}{\partial y}$.

(12)

(c) Find the smallest positive root of equation $xe^x - \cos x = 0$ using regula-falsi method. Do three iterations.

(12)

(d) State the principle of duality

(i) in Boolean algebra and given the dual of the Boolean expressions

$(X+Y)\cdot(\bar{X}\cdot\bar{Z})\cdot(Y+Z)$ and $X\cdot\bar{X}=0$

(6)

(ii) Represent

$(\bar{A}+\bar{B}+\bar{C})(A+\bar{B}+C)(A+B+\bar{C})$ in NOR to NOR logic network

(6)

(e) A circular board is placed on a smooth horizontal plane and a boy runs round the edge of it at a uniform rate. What is the motion of the centre of the board ? Explain. What happens if the mass of the board and boy are equal ?

(12)

(f) If the velocity potential of a fluid is $\phi=\frac{1}{r^3}z.\tan^{-1}\left(\frac{y}{x}\right)$, $r^2 = x^2 + y^2 + z^2$ then show that the stream lines lie on the surfaces $x^2 + y^2 + z^2 = c(x^2 + y^2)^{2/3}$, c being a constant

(12)

6. (a) Find the steady state temperature distribution in a thin rectangular plate bounded by the lines x = 0, x = a, y = 0 and y = b. The edges x = 0, x = a and y = 0 are kept at temperature zero while the edge y = b is kept at 100°C.

(30)

(b) Find complete and singular integrals of

$$2xz - px^2 - 2qxy + pq = 0$$

using Charpit's method.

(15)

(c) Reduce $\frac{\partial^2 z}{\partial x^2}=x^2\frac{\partial^2 z}{\partial y^2}$ to canonical form.

(15)

7. (a) (i) The following values of the function f(x) = sin x + cos x are given:

x	:	10°	20°	30°

f(x) : 1·1585 1·2817 1·3360

Construct the quadratic interpolating polynomial that fits the data. Hence calculate $f\left(\frac{\pi}{12}\right)$. Compare with exact value.

(ii) Apply Gauss-Seidel method to calculate x, y, z from the system:

$$-x-y+6z=42$$
$$6x-y-z=11\cdot33$$
$$-x+6y-z=32$$

with initial values (4·67, 7·62, 9·05). Carry out computations for two iterations.

(15)

(b) Draw a flow chart for solving equation F(x) = 0 correct to five decimal places by Newton-Raphson method.

(30)

8. (a) A uniform rod of mass 3m and length $2l$ has its middle point fixed and a mass m is attached to one of its extremity. The rod, when in a horizontal position is set rotating about a vertical axis through its centre with an angular velocity $\sqrt{\frac{2g}{l}}$. Show that the heavy end of the rod will fall till the inclination of the rod to the vertical is $\cos^{-1}(\sqrt{2}-1)$.

(30)

(b) Let the fluid fills the region $x \geq 0$(right half of 2d plane). Let a source be at $(0, y_1)$ and equal sink at $(0, y_2)$, $y_1 > y_2$. Let the pressure be same as pressure at infinity i.e. p_0. Show that the resultant pressure on the boundary (y-axis) is

$\pi\rho\alpha^2(y_1-y_2)^2/2y_1y_2(y_1+y_2)$,

ρ being the density of the fluid.

(30)