

Time Allowed: 3 hours

Maximum Marks: 300

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

Section-A

1. Attempt any five of the following:
 - (a) Let S be the vector space of all polynomials $p(x)$, with real coefficients, of degree less than or equal to two considered over the real field \mathbb{R} , such that $p(0) = 0$ and $p(1) = 0$. Determine a basis for S and hence its dimension. (12)
 - (b) Let T be the linear transformation from \mathbb{R}^3 to \mathbb{R}^4 defined by
 $T(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 + x_2, x_1 + x_3, 3x_1 + x_2 + 2x_3)$
 For each $(x_1, x_2, x_3) \in \mathbb{R}^3$.
 Determine a basis for the Null space of T . What is the dimension of the Range space of T ? (12)
 - (c) Let $f(x)$ ($x \in (-\pi, \pi)$)? If it is continuous, then it is differentiable on $((-\pi, \pi))$? (12)
 - (d) A figure bounded by one arch of a cycloid
 $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $t \in [0, 2\pi]$, and the x -axis is revolved about the x -axis. Find the volume of the solid of revolution. (12)
 - (e) Find the focus of the point which moves so that its distance from the plane $x + y - z = 1$ is twice its distance from the line $x = -y = z$. (12)
 - (f) Find the equation of the sphere inscribed in the tetrahedron whose faces are $x = 0$, $y = 0$, $z = 0$ and $2x + 3y + 6z = 6$. (12)
2. (a) Let W be the set of all 3×3 symmetric matrices over \mathbb{R} . Does it form a subspace of the vector space of the 3×3 matrices over \mathbb{R} ? In case it does, construct a basis for this space and determine its dimension. (20)
- (b) Consider the vector space
 $X = \{p(x) \mid p(x) \text{ is a polynomial of degree less than or equal to 3 with real coefficients}\}$, over the real field \mathbb{R} . Define the map $D : X \rightarrow X$ by

$$(Dp)(x) : p_1 + 2p_2x + 3p_3x^2$$

$$\text{where } p(x) = p_0 + p_1x + p_2x^2 + p_3x^3 +$$

Is D a linear transformation on X? If it is, then construct the matrix representation for D with respect to the ordered basis $\{1, x, x^2, x^3\}$ for X.

(20)

- (c) Reduce the quadratic form $q(x, y, z) = x^2 + 2y^2 - 4xz + 4yz + 7z^2$ to canonical form. Is q positive definite?

(20)

3. (a) Find a rectangular parallelopiped of greatest volume for a given total surface area S, using Lagrange's method of multipliers.

(20)

- (b) Prove that if $z = \psi(y + ax) + \psi(y - ax)$ then

$$a^2 \frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} = 0$$

for any twice differentiable ϕ and ψ ; a is a constant.

(15)

- (c) Show that $e^{-x} x^n$ is bounded on $[0, \infty)$ for all positive integral values of n. Using this result show that

$$\int_0^\infty e^{-x} x^n dx \text{ exists.}$$

(25)

4. (a) Show that the spheres $x^2 + y^2 + z^2 - x + z - 2 = 0$ and $3x^2 + 3y^2 - 8x - 10y + 8z + 14 = 0$ cut orthogonally. Find the centre and radius of their common circle.

(15)

- (b) A line with direction ratios 2, 7, -5 is drawn to intersect the lines

$$\frac{x}{3} = \frac{y-1}{2} = \frac{z-2}{4} \text{ and } \frac{x-11}{3} = \frac{y-5}{-1} = \frac{z}{1}$$

Find the coordinates of the points of intersection and the length intercepted on it.

(15)

- (c) Show that the plane $2x - y + 2z = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines.

(15)

- (d) Show that the feet of the normals from the point $P(\alpha, \beta, \gamma)$, $\beta \neq 0$ on the paraboloid $x^2 + y^2 = 4z$ lie on the sphere

$$2\beta(x^2 + y^2 + z^2) - (\alpha^2 + \beta^2)y - 2\beta(2 + \gamma)z = 0$$

(15)

SECTION B

5. Answer any FIVE of the following:

- (a) (i) Form a partial differential equation by eliminating the function f from :

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

(6)

- (ii) Solve

$$2zx - px^2 - 2qyx + pq = 0.$$

(6)

- (b) Transform the equation

$$yz_x - xz_y = 0$$

into one in polar coordinates and thereby show that the solution of the given equation represents surfaces of revolution.

(12)

- (c) Use the method of false position to find a real root of $x^3 - 5x - 7 = 0$ lying between 2 and 3 and correct to 3 places of decimals.

(12)

- (d) Convert :

- (i) 46655 given to be in the decimal system into one in base 6,

- (ii) $(11110.01)_2$ into a number in the decimal system.

(6+6)

- (e) Consider a system with two degree of freedom for which

$$V = q_1^2 + 3q_1q_2 + 4q_2^2.$$

Find the equilibrium position and determine if the equilibrium is stable.

(12)

- (f) Show that $\left(\frac{x^2}{a^2}\right)\cos^2 t + \left(\frac{y^2}{b^2}\right)\sec^2 t = 1$ is a possible form for the boundary surface of a liquid.

(12)

6. (a) Solve

$$u_{xx} + u_{yy} = 0 \text{ in } D$$

where $D = \{(x, y) : 0 < x < a, 0 < y < b\}$ is a rectangle in a plane with the boundary conditions:

$$u(x, 0) = 0, u(x, b) = 0, 0 \leq x \leq a$$

$$u(0, y) = g(y), u_x(a, y) = h(y), 0 \leq y \leq b.$$

(30)

7. (a) A particle is performing simple harmonic motion of period T about a centre O . It passes through a point P ($OP = p$) with velocity v in the direction OP . Show that the time which elapses before it returns to P is

$$\frac{T}{\pi} \tan^{-1} \frac{vT}{2\pi p}$$

- (b) A particle attached to a fixed peg O by a string of length 1, is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is $g\sqrt{(1+3\sin^2\theta)}$

(15)

- (c) A uniform beam of length 1 rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β ($\beta > \alpha$), show that the inclination θ of the beam to the horizontal, in one of the equilibrium positions, is given by

$$\tan \theta = \frac{1}{2}(\cot \alpha - \cot \beta)$$

& show that the beam is unstable in this position.

(15)

- (d) A cone whose vertical angle is $\frac{\pi}{3}$, has its lowest generator horizontal and is filled with a liquid. Prove that the pressure on the curved surface is $\frac{W}{2}\sqrt{19}$ where W is the weight of the liquid.

(15)

8. (a) Find the curvature and torsion at any point of the curve $x = a \cos 2t$, $y = a \sin 2t$, $z = 2a \sin t$.

(15)

- (b) For any constant vector \vec{a} , show that the vector represented by $\text{curl}(\vec{a} \times \vec{r})$ is always parallel to the vector \vec{a} , \vec{r} being the position vector of a point (x, y, z) measured from the origin.

(15)

- (c) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find the value(s) of n in order that $r^n \vec{r}$ may be (i) solenoidal, (ii) irrotational.

(15)

- (d) Determine $\int_C (ydx + zdy + xdz)$ by using Stoke's theorem, where C is the curve defined by

$$(x-a)^2 + (y-a)^2 + z^2 = 2a^2, x+y = 2a$$

that starts from the point $(2a, 0, 0)$ and goes at first below the z -plane.

(15)