# IAS
# PREVIOUS YEARS QUESTIONS (2017-1983)
## SEGMENT-WISE

## COMPLEX ANALYSIS

### 2017

- Using contour integral method, prove that

$$\int_0^\infty \frac{x\sin mx}{a^2+x^2}dx = \frac{\pi}{2}e^{-ma}$$ **(15)**

- For a function $f:\mathbb{C}\to\mathbb{C}$ and $n \ge 1$, let $f^{(n)}$ denote the $n^{th}$ derivative of f and $f^{(0)} = f$. Let f be an entire function such that for some $n \ge 1,\ f^{(n)}\left(\frac{1}{k}\right)=0$ for all k = 1, 2, 3,....... Show that f is a polynomial. **(15)**

- Let f = u + iv be an analytic function on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$$ at all points of D. **(15)**

- Determine all entire functions f(z) such that 0 is a removable singularity of $f\left(\frac{1}{z}\right)$. **(10)**

### 2016

- Is $v(x, y) = x^3 - 3xy^2 + 2y$ a harmonic function ? Prove your claim. If yes, find its conjugate harmonic function u(x, y) and hence obtain the analytic function whose real and imaginary parts are u and v respectively. **(10)**

- Let $\gamma : [0,1] \to \mathbb{C}$ be the curve

$\gamma(t) = e^{2\pi i t}, 0 \le t \le 1$.

Find, giving justifications, the value of the contour integral

$$\int_\gamma \frac{dz}{4z^2-1}$$ **(15)**

- Prove that every power series represents an analytic function inside its circle of convergence. **(20)**

### 2015

- Show that the function $v(x, y) = \ln(x^2 + y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x, y)$. Also find the corresoponding analytic function $f(z) = u + iv$ in terms of $z$.

- Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \frac{2z-3}{z^2-3z+2}$ about the point $z = 0$.

- State Cauchy's residue theorem. Using it, evaluate the integral $\int_C \frac{e^z+1}{z(z+1)(z-i)^2}dz; C : |z| = 2$.

### 2014

- Prove that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, z \ne 0; f(0) = 0$$

satisfies Cauchy-Riemann equations at the origin, but the derivative of $f$ at $z = 0$ does not exist.

- Expand in Laurent series the function

$$f(z) = \frac{1}{z^2(z-1)}$$ about z = 0 and z = 1.

- Evaluate the integral $\int_0^\pi \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$ using residues.

### 2013

- Prove that if $b\,e^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - b\,e^z$ has $n$ zeroes in the unit circle.

- Using Cauchy's residue theorem, evaluate the integral

$$I = \int_0^\pi \sin^4\theta\, d\theta$$

## 2012

- Show that the function defined by

$$f(z)=\begin{cases}\dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z\neq 0\\ 0, & z=0\end{cases}$$

is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin. **(12)**

- Use Cauchy integral formula to evaluate $\int_C \frac{e^{3z}}{(z+1)^4}dz,$ where $C$ is the circle $|z|=2.$ **(15)**

- Expand the function $f(z)=\frac{1}{(z+1)(z+3)}$ in Laurent series valid for

(i) $1<|z|<3$ (ii) $|z|>3$

(iii) $0<|z+1|<2$ (iv) $|z|<1$ **(15)**

- Evaluate by Contour integration

$$I=\int_0^{2\pi}\frac{d\theta}{1-2a\cos\theta+a^2}, a^2<1.$$ **(15)**

## 2011

- Evaluate by Contour integration, $\int_0^1 \frac{dx}{\left(x^2-x^3\right)^{1/3}}$. **(15)**

- Find the Laurent Series for the function

$f(z)=\frac{1}{1-z^2}$ with centre z=1. **(15)**

- Show that the series for which the sum of first n terms $f_n(x)=\frac{nx}{1+n^2x^2}, 0\le x\le 1$ cannot be differentiated term-by-term at x=0.What happens at $x\neq 0$? **(15)**

- If f(z)=u+iv is an analytic function of z=x+iy and $u-v=\frac{e^y-\cos x+\sin x}{\cos hy-\cos x}$ ,find f(z) subject to the condition, $f\left(\frac{\pi}{2}\right)=\frac{3-i}{2}.$ **(12)**

## 2010

- Show that $u(x,y)=2x-x^3+3xy^2$ is a harmonic function. Find a harmonic conjugate of u(x, y). Hence find the analytic function f for which u(x, y) is the real part. **(12)**

- (i) Evaluate the line integral $\int_c f(z)dz$. Where $f(z)=z^2$, c is the boundary of the triangle with vertices A (0, 0), B (1, 0), C (1, 2) in that order.

  (ii) Find the image of the finite vertical strip R : x = 5 to x =9, -π ≤ y ≤ π of z – plane under the exponential function. **(15)**

- Find the Laurent series of the function

$$f(z)=\exp\left[\frac{\lambda}{2}\left(z-\frac{1}{z}\right)\right] as \sum_{n=-\infty}^{\infty}c_nz^n \quad for\ 0<|z|<\infty$$

Where $C_n=\frac{1}{\pi}\int_0^{\pi}\cos(n\phi-\lambda\sin\phi)d\phi, n=0,\pm1,\pm2,\ldots,$

with λ a given complex number and taking the unit circle C given by $z=e^{i\phi}(-\pi\le\phi\le\pi)$ as contour in this region. **(15)**

## 2009

- Let $f(z)=\frac{a_0+a_1z+.......+a_{n-1}z^{n-1}}{b_0+b_1z+.......+b_nz^n}, b_n\neq 0.,$

Assume that the zeroes of the denominator are simple. Show that the sum of the residues of f(z) at its poles is equal to $\frac{a_{n-1}}{b_n}$. **(12)**

- If α, β, γ are real numbers such that $\alpha^2>\beta^2+\gamma^2$

Show that: $\int_0^{2\pi}\frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta}=\frac{2\pi}{\sqrt{\alpha^2-\beta^2-\gamma^2}}$ **(30)**

## 2008

- Find the residue of $\frac{\cot z\cot hz}{z^3}$ at z = 0. **(12)**

- Evaluate $\int_C \left[\frac{e^{2z}}{z^2(z^2+2z+2)}+\log(z-6)+\frac{1}{(z-4)^2}\right]dz$

  where c is the circle |Z|=3. State the theorem you use in evaluating above integral. **(15)**

- Let f(z) be entire function satisfying $f(z) \le k|z|^2$ for some +ve constant $k$ and all $z$. show that $f(z)=az^2$ for some constant a. **(15)**

## 2007

- Prove that the function f defined by

  $$f(z)=\begin{cases}\frac{z^5}{|z|^4}, z\neq 0\\ 0,\ z=0\end{cases}$$ is not differentiable at z=0 **(12)**

- Evaluate (by using residue theorem) $\int_0^{2\pi}\frac{d\theta}{1+8\cos^2\theta}$. **(15)**

## 2006

- With the aid of residues, evaluate

  $\int_0^{\pi}\frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2}; -1<a<1.$ **(15)**

- If f (z) = u + iv is an analytic function of the complex variable z and $u-v=e^x(\cos y-\sin y)$ determine f(z) in terms of z. **(12)**

- Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in Laurent's series which is valid for (i) 1<|z|<3 (ii) |z| >3 (iii) |z| < 1. **(30)**

## 2004

- If all zeros of a polynomial p(z) lie in a half plane then show that zeros of the derivative p'(z) also lie in the same half plane. **(15)**
- Using contour integration , evaluate

  $\int_0^{2\pi}\frac{\cos^2 3\theta d\theta}{1-2p\cos 2\theta+p^2}, 0<p<1.$ **(15)**

## 2003

- Use the method of contour integration to prove that $\int_0^{\pi}\frac{a d\theta}{a^2+\sin^2\theta}=\frac{\pi}{\sqrt{1+a^2}};(a>0).$ **(15)**

## 2002

- Suppose that f and g are two analytic functions on the set $\mathbb{C}$ of all complex numbers with $f\left(\frac{1}{n}\right)=g\left(\frac{1}{n}\right)$ for n=1,2,3,….. then show that f(z)=g(z) for each z in $\mathbb{C}$ . **(12)**
- Show that when 0 < | z-1 | < 2, the function $f(z)=\frac{z}{(z-1)(z-3)}$ has the Laurent series expansion in powers of z-1 as

  $-\frac{1}{2(z-1)}-3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}.$ **(15)**

## 2001

- Prove that the Riemann Zeta function $\xi$ defined by $\xi(z)=\sum_{n=1}^{\infty}n^{-z}$ converges for $\operatorname{Re} z>1$ and converges uniformly for $\operatorname{Re} z\ge 1+\in$ where $\in>0$ is arbitrary small. **(12)**

- Show that $\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx=\frac{\pi}{\sqrt{2}}$.

## 2000

- Suppose $f(\xi)$ is continuous on a circle C. show that $\int_C\frac{f(\xi)}{(\xi-z)}d\xi$ as z varies inside of 'C', is differentiable under the integral sign. Find the derivative hence or otherwise derive an integral representation for $f'(z)$ if $f(z)$ is analytic on and inside of C. **(30)**

## 1999

- Examine the nature of the function $f(z)=\frac{x^2y^5(x+iy)}{x^4+y^{10}}, z\neq 0$ $f(0)=0$ in a region including the origin and hence show that Cauchy – Riemann equations are satisfied at the origin but f(z) is not analytic there.

- For the function $f(z)=\frac{-1}{z^2-3z+2}$, find Laurent series for the domain (i) $1<|z|<2$ (ii) $|z|>2$ show further that $\oint_c f(z)\,dz=0$ where 'c' is any closed contour enclosing the points z=1 and z=2 .
- Using residue theorem show that $\int_{-\infty}^{\infty}\frac{x\sin ax}{x^4+4}dx=\frac{\pi}{2}e^{-a}\sin a;(a>0)$ ***(1984, 1998)***
- The function f(z) has a double pole at z=0 with residue 2, a simple pole at z=1 with residue 2, is analytic at all other finite points of the plane and is bounded as $|z|\to\infty$. If f(2)=5 and f(-1)=2 , find f(z).
- What kind of singularities the following functions have?

  (i) $\frac{1}{1-e^z}$ at $z=2\pi i$

  (ii) $\frac{1}{\sin z-\cos z}$ at $z=\frac{\pi}{4}$

  (iii) $\frac{\cot\pi z}{(z-a)^2}$ at $z=a$ and $z=\infty$.

  In case (iii) above what happens when 'a' is an integer.(including a=0)?

## 1998

- Show that the function

  $$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, z\neq 0$$

- $f(0)=0$ is continuous and C-R conditions are satisfied at z=0, but $f'(z)$ does not exist at z=0.
- Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity z =–2. Specify the region of convergence and the nature of singularity at z = -2
- By using the integral representation of $f^n(0)$, prove that $\left(\frac{x^n}{n!}\right)^2=\frac{1}{2\pi i}\oint_c\frac{x^n e^{xz}}{n!z^{n+1}}dz$, where 'c' is any closed contour surrounding the origin. Hence show that $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^2=\frac{1}{2\pi}\int_0^{2\pi}e^{2x\cos\theta}d\theta.$
- Using residue theorem $\int_0^{2\pi}\frac{d\theta}{3-2\cos\theta+\sin\theta}$.

## 1997

- If $f(z)=\frac{A_1}{z-a}+\frac{A_2}{(z-a)^2}+-----+\frac{A_n}{(z-a)^n}$ find the residue at a for $\frac{f(z)}{z-b}$ where $A_1, A_2, ..........A_n$, a & b are constant. What is the residue at infinity.
- Find the Laurent series for the function $e^{1/z}$ in $0<|z|\le\infty$.

  Deduce that $\frac{1}{\pi}\int_0^{\pi}e^{\cos\theta}\cos(\sin\theta-n\theta)d\theta=\frac{1}{n!}$

  $(n=0,1,2,..........)$ **(2001)**
- Find the function f(z) analytic with in the unit circle which takes the values

  $\frac{a-\cos\theta+i\sin\theta}{a^2-2a\cos\theta+1}, 0\le\theta\le 2\pi$ on the circle.
- Integrating $e^{-z^2}$ along a suitable rectangular contour. Show that $\int_0^{\infty}e^{-x^2}\cos bx\,dx=\frac{\sqrt{\pi}}{2}e^{-b^2}$.

## 1996

- Evaluate $\underset{z\to 0}{Lt}\frac{1-\cos z}{\sin(z^2)}$
- Show that z=0 is not a branch point for the function $f(z)=\frac{\sin\sqrt{z}}{\sqrt{z}}$. Is it a removable singularity ?
- Prove that every polynomial equation $a_0+a_1z+a_2z^2+.......+a_nz^n=0, a_n\neq 0, n\ge 1$ has

exactly 'n' roots.

- By using residue theorem, evaluate $\int_0^\infty \frac{\log_e\left(x^2+1\right)}{x^2+1}dx$
- About the singularity $z=-2$, find the Laurent expansion of $(z-3)\sin\frac{1}{z+2}$. Specify the region of convergence and nature of singularity at $z=-2$.

## 1995

- Let $u(x,y)=3x^2y+2x^2-y^3-2y^2$. Prove that 'u' is a harmonic function. Find a harmonic function v such that u+iv is an analytic function of z.
- Find the Taylor series expansion of the function $f(z)=\frac{z}{z^4+9}$ around z = 0. Find also the radius of convergence of the obtained series.
- Let 'C' be the circle | Z |=2 described contour clockwise. Evaluate the integral $\int_c \frac{\cosh \pi z}{z\left(z^2+1\right)}dz$
- Let $a \geq 0$. Evaluate the integral $\int_0^\infty \frac{\cos ax}{x^2+1}dx$ with the aid of residues. ***(2006)***
- Let f be analytic in the entire complex plane. Suppose that there exists a constant A > 0 such that $|f(z)| \leq A|z|$ for all z. Prove that there exists a complex number 'a' such that f(z)=az for all z.
- Suppose a power series $\sum_{n=0}^{\infty} a_n z^n$ converges at a point $z_0 \neq 0$.

  Let $z_1$ be such that $|z_1|<|z_0|$ and $z_1 \neq 0$ $|z_1|<|z_0|$ and $z_1 \neq 0$ show that the series converges uniformly in the disc $\{z:|z|\leq|z_1|\}$.

## 1994

- How many zeros does the polynomial $p(z)=z^4+2z^3+3z+4$. Posses in (i) the first quadrant (ii) the fourth quadrant.
- Test for uniform convergence in the region $|Z|\leq 1$ the series $\sum_{n=1}^{\infty}\frac{\cos nz}{n^3}$.
- Find Laurent series for (i) $\frac{e^{2z}}{(z-1)^3}$ about z=1.(ii) $\frac{1}{z^2(z-3)^2}$ about $z=3$.
- Find the residues of $f(z)=e^z \operatorname{cosec}^2 z$ at all its poles in the finite plane.
- By means of contour integration, evaluate $\int_0^\infty \frac{(\log_e u)^2}{u^2+1}du$.

## 1993

- In the finite Z- plane show that the function

  $f(z)=\sec\frac{1}{z}.$

  has infinitely many isolated singularities in a finite interval which includes '0'.
- Prove that (by applying Cauchy integral formula or otherwise)

  $$\int_0^{2\pi}\cos^{2n}\theta d\theta=\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}2\pi,$$

  *where* $n=1,2,3,\ldots$
- If C is the curve $y=x^3-3x^2+4x-1$ joining the points (1,1) and (2,3) find the value of $\int_C (12z^2-4iz)\,dz$
- Prove that $\sum_{n=1}^{\infty}\frac{z^n}{n(n+1)}$ converges absolutely for $|z|\leq 1$.
- Evaluate $\int_0^\infty \frac{dx}{x^6+1}$ by choosing an appropriate contour.

## 1992

- If $u=e^{-x}$ (x siny-ycosy), find 'v' such that f(z)= u+iv is analytic. Also find f(z) explicitly as a function of z. **(1997)**

- Let f(z) be analytic inside and on the circle C defined by |z| = R and let $z = re^{i\theta}$ be any point inside C. prove that

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(R^2 - r^2\right) f\left(\mathrm{Re}^{i\phi}\right)}{R^2 - 2Rr\cos\left(\theta - \phi\right) + r^2} d\phi .$$

- Prove that all roots of $z^7 - 5z^3 + 12 = 0$ lies between the circles $|z| = 1$ *and* $|z| = 2$ . **(1998,2006)**
- Find the region of convergence of the series whose n-th term is $\frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$
- Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for

$(i)|z| > 3 \quad (ii) 1 < |z| < 3 \quad (iii)|z| < 1$ **(2005)**

- By integrating along a suitable contour evaluate $\int_0^\infty \frac{\cos mx}{x^2+1} dx$ .

## 1991

- **A** function f(z) is defined for finite values of z by f(0)=0 and $f(z) = e^{-z^{-4}}$ everywhere else. Show that the Cauchy Riemann equation are satisfied at the origin. Show also that f(z) is not analytic at the origin.
- If |a| ≠ R show that $\int_{|z|=R} \frac{|dz|}{|z-a||z+a|} < \frac{2\pi R}{\left|R^2 - |a|^2\right|}$
- If $J_n(t) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - t\sin\theta) d\theta .$ show that

$$e^{\frac{1}{2}\left(z - \frac{1}{z}\right)} = J_0(t) + zJ_1(t) + z^2 J_2(t) + ---$$

$$-\frac{1}{z}J_1(t) + \frac{1}{z^2}J_2(t) - \frac{1}{z^3}J_3(t) + ---$$

- Examine the nature of the singularity of $e^z$ at infinity
- Evaluate the residues of the function $\frac{z^3}{(Z-2)(Z-3)(Z-5)}$ at all singularities and show that their sum is zero.
- By integrating along a suitable contour show that $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} = \frac{\pi}{\sin a\pi}$ where 0 < a < 1.

## 1990

- Let f be regular for |Z| < R , prove that , if 0<r<R, $f'(0) = \frac{1}{\pi r}\int_0^{2\pi} u(\theta) e^{-i\theta} d\theta$ ; where $u(\theta) = \mathrm{Re}\ f(re^{i\theta})$
- Prove that the distance from the origin to the nearest zero of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is at least $\frac{r|a_0|}{M + |a_0|}$ . where r is any number not exceeding the radius of the convergence of the series and $M = M(r) = \sup_{|z|=r} |f(z)|$.
- Prove that $\int_{-\infty}^{\infty} \frac{x^4}{1+x^8} dx = \frac{\pi}{\sqrt{2}} \sin\frac{\pi}{8}$ using residue calculus.
- Prove that if f=u+iv is regular through out the complex plane and au+bv-c ≥0 for suitable constants a,b,c then f is constant.
- Derive a series expansion of $\log(1+e^z)$ in powers of z.
- Determine the nature of singular points $\sin\left(\frac{1}{\cos 1/z}\right)$ and investigate its behaviour at $z = \infty$ .

## 1989

- Find the singularities of $\sin\left(\frac{1}{1-z}\right)$ in the complex plane.

## 1988

- By evaluating $\int \frac{dz}{z+2}$ over a suitable contour C, Prove that $\int_0^\pi \frac{1+2\cos\theta}{5+4\cos\theta} d\theta$ ***(1997)***

❖ If $f$ is analytic in $|Z| \leq$ R and x, y lie inside the disc, evaluate the integral $\int_{|z|=R} \frac{f(z)dz}{(z-x)(z-y)}$ and deduce that a function analytic and bounded for all finite z is a constant.

❖ If $f(z)=\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R and prove that $\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$

❖ Evaluate $\int_C \frac{Ze^z}{(z-a)^3}$ , if a lies inside the closed contour C.

❖ Prove that $\int_0^{\infty} e^{-x^2}\cos(2bx)\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2};(b>0)$ by the integrating $e^{-Z^2}$ along the boundary of the rectangle |x| d" R,0 d" y d"b. **(1997)**

❖ Prove that the coefficients $C_n$ of the expansion $\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} C_n z^n$ satisfy $C_n = C_{n-1} + C_{n-2}$ ,$n \geq 2$ Determine $C_n$.

## 1987

❖ By considering the Laurent series for $f(z) = \frac{1}{(1-z)(z-2)}$ prove that if 'C' be a closed contour oriented in the contour clockwise direction, then $\int_C f(z)\,dx = 2\pi i$

❖ State and prove Cauchy's residue theorem

❖ By the method of contour integration , show that $\int_0^{\infty} \frac{\cos x}{x^2+a^2}dx = \frac{\pi e^{-x}}{2a}, a > 0.$

## 1986

❖ Let f(z) be single valued and analytic with in and on a closed curve C. If $z_0$ is any point interior to C, then show that $f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}dz$ ,where the integral is taken in the +ve sense around C.

❖ By contour integration method show that

(i) $\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$ ,where $a > 0$.

(ii) $\int_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$.

## 1985

❖ Prove that every power series represents an analytic function within its circle of convergence.

❖ Prove that the derivative of a function analytic in a domain is itself an analytic function.

❖ Evaluate , by the method of contour integration $\int_0^{\infty} \frac{x\sin ax}{x^2-b^2}dx$.

## 1984

❖ Evaluate by contour integration method :

(i) $\int_0^{\infty} \frac{x\sin mx}{x^4+a^4}dx$

(ii) $\int_0^{\infty} \frac{x^{a-1}\log x}{1+x^2}dx$ **(1998, 1999)**

❖ Distinguish clearly between a pole and an essential singularity. If z=a is an essential singularity of a function f(z) , then for an arbitrary positive integers $\eta, \in and\ \rho$ , prove that $\exists$ a point z , such that $0 < |z-a| < \rho$ $for\ which \left|f(z)-\eta\right| < \in$.

## 1983

❖ Obtain the Taylor and Laurent series expansions which represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the regions (i)|z|<2 (ii) 2<|z|<3 (iii) |z|>3.

❖ Use the method of contour integration to evaluate $\int_0^{\infty} \frac{x^{a-1}}{1+x^2}dx, 0 < a < 2$.