TS- Hyd

163/250

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

Hyd

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-I(M) IAS / T- 05

MATHEMATICS

by K. VENKANNA

The person with 14 years of Teaching Experience

FULL TEST P-I

Time: Three Hours | *Maximum Marks: 250*

INSTRUCTIONS

1. This question paper-cum-answer booklet has 51 pages and has 33 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Varun Guntupalli

Roll No.:

Test Centre: Hyderabad

Medium: English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

Q.No.	Total Marks	Q.No.	Total Marks
1 a.		4 c.	
1 b.	8 -	4 d.	
1 c.		5 a.	8 -
1 d.	8 -	5 b.	8 -
1 e.	8 -	5 c.	
2 a.		5 d.	
2 b.		5 e.	7
2 c.		6 a.	8 -
2 d.		6 b.	11
3 a.	18	6 c.	10
3 b.	13	6 d.	12
3 c.	13	7 a.	
3 d.		7 b.	
4 a.		7 c.	
4 b.		7 d.	
		8 a.	10
		8 b.	6
		8 c.	10
		8 d.	5

P.T.O.

SECTION-A

1. (a) Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into: $det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)......(\lambda_n-\lambda)$ and making a clever choice of λ. **(10)**

1. (b) If $A=\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, then det $(A-\lambda I)$ is $(\lambda-a)(\lambda-d)$. Check the Cayley-Hamilton statement that $(A-aI)(A-dI) = zero matrix.$ (10)

Given $\det(A-\lambda I) = (\lambda-a)\cdot(\lambda-d)$.

i.e. $\begin{vmatrix} a-\lambda & b \\ 0 & d-\lambda \end{vmatrix} = (\lambda-a)(\lambda-d)$ which is true.

LHS of Cayley Hamilton statement

$$= (A-aI)(A-dI)$$

$$= \left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right\} \cdot \left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} - \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix} \cdot \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0\cdot(a-d)+b\cdot 0 & 0\cdot b+b\cdot 0 \\ 0\cdot(a-d)+(d-a)\cdot 0 & 0\cdot b+(d-a)\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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$$= \text{zero matrix.}$$

Hence, the Cayley-Hamilton statement is verified.

1. (c) Show that $\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$ isconvergent and hence evaluate it. **(10)**

$$I = \int_0^{\pi/2} \log(\sin x)\,dx$$

~~We know that~~

$$\cancel{\log y = y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots}$$

~~We know that $\log y < y$~~

$$\cancel{\Rightarrow I = \int_0^{\pi/2} \log(\sin x)\,dx < \int_0^{\pi/2} \cos x\,dx = \sin x\Big|_0^{\pi/2}}$$
$$\cancel{= 1}$$

$$\cancel{\Rightarrow I < 1}$$

We know that

$$\log y = y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots$$

$$\Rightarrow I = \int_0^{\pi/2} \left(\sin x + \frac{1}{2!}\sin^2 x + \frac{1}{3!}\sin^3 x + \cdots\right) dx$$

$$= 1 + \frac{1}{2!}\left(\frac{\pi}{2}\right) + \frac{1}{3!}\Big($$

1. (d) The plane $lx+my=0$ is rotated about the line of intersection with the plane $z=0$ through an angle α. Prove that the equation to the plane in its new position is **(10)**

$$lx+my\pm z\sqrt{(l^2+m^2)}\tan\alpha=0$$

Any plane through $lx+my=0$ —(1) & $z=0$ —(2) is

given by $lx+my+\lambda z=0$ —(3)

If (3) is the required plane, angle between

(3) & (1) is α

$$\Rightarrow \cos\alpha = \frac{l\cdot l+m\cdot m}{\sqrt{l^2+m^2}\cdot\sqrt{l^2+m^2+\lambda^2}}$$

$$\Rightarrow \cos^2\alpha = \frac{l^2+m^2}{l^2+m^2+\lambda^2}$$

$$\Rightarrow \lambda^2(\cos^2\alpha) = (l^2+m^2)(1-\cos^2\alpha)$$

$$\Rightarrow \lambda^2 = (l^2+m^2)\cdot\tan^2\alpha$$

$$\Rightarrow \lambda = \pm\sqrt{l^2+m^2}\,\tan\alpha \quad — (4)$$

Put (4) in (3) to get the equation of required plane as

$$lx+my\pm\sqrt{l^2+m^2}\,\tan\alpha\cdot z = 0$$

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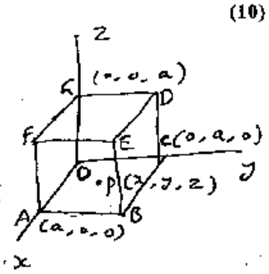
1. (e) A point moves so that the sum of the squares of its distances from the six faces of a cube is constant, show that its locus is a sphere. **(10)**



Let the cube be having origin as one of the vertices & 3 edges of the cube along the x, y, z axes & let side of the cube $= a$.

Let $P(x, y, z)$ be any point

Foot of $\perp$rs from P to each of the faces are given by

~~OAFG~~ OABD	$(x, y, 0)$
FEDG	(x, y, a)
OAFG	$(x, 0, z)$
BCDE	(x, a, z)
OCDG	$(0, y, z)$
ABEF	(a, y, z)

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$\Rightarrow$ Sum of squares of distances of P. from 6 faces = k, a constant.

$\Rightarrow \quad z^2 + (z-a)^2 + y^2 + (y-a)^2 + x^2 + (x-a)^2 = k$

$\Rightarrow \quad 2x^2 + 2y^2 + 2z^2 - 2ax - 2ay - 2az + 3a^2 = k$

which is the equation of a sphere with centre (a, a, a)

i.e. the ~~seqmt~~ locus is a sphere.

2. (a) Find the rank and the nullspace of **(10)**

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

3. (a) (i) Find the largest possible number of independent vectors among

$$v_1=\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\quad v_2=\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}\quad v_3=\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\quad v_4=\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\quad v_5=\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\quad v_6=\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}.$$

(ii) Do the vectors (1, 1, 3), (2, 3, 6) and (1, 4, 3) form a basis for $\mathbf{R}^3$?

(20)

(i) Clearly, as all V_is are elements of $\mathbb{R}^4$, the maximum no. of independent vectors among them = 4.

Clearly V_1, V_2, V_3 are linearly independent

$\because$ If $\lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3 = 0$ then $\lambda_1 = \lambda_2 = \lambda_3 = 0$

Now consider V_1, V_2, V_3, V_4,

we have $V_4 = -V_1 + V_2$

Now consider V_1, V_2, V_3, V_5,

we have $V_5 = -V_1 + V_3$

Now consider V_1, V_2, V_3, V_5,

we have $V_6 = -V_2 + V_3$

please consult key for other method

~~$\Rightarrow V_4, V_5, V_6$ are~~

Also if we consider V_4, V_5, V_6 &

if $\lambda_1 V_4 + \lambda_2 V_5 + \lambda_3 V_6 = 0$

we have $\lambda_1 + \lambda_2 = 0$

$-\lambda_1 + \lambda_3 = 0 \quad \Rightarrow \quad \lambda_1 = -\lambda_2 = \lambda_3$

$-\lambda_2 + \lambda_3 = 0$

$\Rightarrow V_4, V_5, V_6$ are linearly dependent.

~~He~~

Hence, largest no. of independent vectors among the given 6 vectors $= 3$ which are v_1, v_2, v_3

(ii) Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$\lambda_1(1,1,3) + \lambda_2(2,3,6) + \lambda_3(1,4,3) = 0$$

i.e.
$$\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \quad -(1)$$
$$\lambda_1 + 3\lambda_2 + 4\lambda_3 = 0 \quad -(2)$$
$$3\lambda_1 + 6\lambda_2 + 3\lambda_3 = 0 \quad -(3)$$

From (1) & (2), $\lambda_2 + 3\lambda_3 = 0 \quad -(4)$

From (2) & (3), $3\lambda_2 + 9\lambda_3 = 0 \quad -(5)$

$\Rightarrow$ From (4) & (5), $\lambda_2 = -3\lambda_3 \quad -(6)$

$\Rightarrow$ From (1) & (6), $\lambda_1 = 5\lambda_3 \quad -(7)$

$\Rightarrow -\frac{3}{5}\lambda_1 = \lambda_2 = -3\lambda_3$

$\Rightarrow \frac{\lambda_1}{5} = \frac{\lambda_2}{-3} = \frac{\lambda_3}{1}$

$\Rightarrow (\lambda_1, \lambda_2, \lambda_3) = (5k, -3k, k)$ where $k \in \mathbb{R}$

5/8 $\Rightarrow$ Given 3 vectors are linearly dependent

$\Rightarrow$ Given vectors do not form a basis for $\mathbb{R}^3$.

3. (b) Find the shortest distance from the origin to the hyperbola $x^2+8xy+7y^2=225$, $z=0$. (15)

Let $P(x, y, 0)$ be a point on the hyperbola

$\Rightarrow$ Distance between P & $O(0,0,0) = \sqrt{x^2+y^2}$

i.e. We'll have to minimize

$f(x,y) = x^2+y^2$ subject to $x^2+8xy+7y^2=225$ — (1)

Applying Lagrange's method, Consider

$g(x,y) = x^2+y^2+\lambda(x^2+8xy+7y^2-225)$

$\frac{\partial g}{\partial x} = 0 \Rightarrow 2x+\lambda(2x+8y) = 0$ — (2)

$\frac{\partial g}{\partial y} = 0 \Rightarrow 2y+\lambda(8x+14y) = 0$ — (3)

$\frac{\partial g}{\partial \lambda} = 0 \Rightarrow x^2+8xy+7y^2-225 = 0$ — (4)

From (2) & (3), $(\lambda+1)x + 4\lambda y = 0$

$4\lambda x + (7\lambda+1)y = 0$

$\Rightarrow \frac{x}{y} = \frac{-4\lambda}{\lambda+1} = \frac{-(7\lambda+1)}{4\lambda}$

$\Rightarrow 16\lambda^2 = (\lambda+1)(7\lambda+1) = 7\lambda^2+8\lambda+1$

$\Rightarrow 9\lambda^2 - 8\lambda - 1 = 0$

$\Rightarrow (\lambda-1)(9\lambda+1) = 0$

$\Rightarrow \lambda = 1, -1/9$

For $\lambda = 1$, $2x+4y=0$ i.e. $x=-2y$

put this in $x^2+8xy+7y^2=225$

i.e. $4y^2-16y^2+7y^2=225$

$\Rightarrow y^2 = -\frac{225}{5}$ $\Rightarrow y$ does not exist.

For $\lambda = -1/9$, $\frac{8}{9}x = -\frac{4}{9}y \Rightarrow y = 2x$

Put this in $x^2 + 8xy + 7y^2 = 225$

$\Rightarrow x^2 + 16x^2 + 7 \times 4x^2 = 225$

$\Rightarrow x^2 = \frac{225}{45} = \frac{15}{3} = 5$

$\Rightarrow y^2 = 4x^2 = 20.$

i.e. $(x, y) = (\sqrt{5}, 2\sqrt{5})$ or $(-\sqrt{5}, -2\sqrt{5})$

& the shortest distance

$= \sqrt{x^2+y^2} = \sqrt{5+20} = \underline{5}$

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3. (c) Show that the plane $x+y-z=0$ cuts the conicoid $4x^2+2y^2+z^2+3yz+zx-1=0$ in a circle. Find also the radius of this circle. (15)

$4x^2+2y^2+z^2+3yz+zx-1=0$ — (1)

$x+y-z=0$ — (2)

~~Using (2) in (1), we can get~~

~~From (1),~~

~~$2x^2+\frac{1}{2}y^2+\frac{1}{2}z^2+\frac{3}{2}(x^2+2yz+$~~

From (1),

$\frac{7}{2}x^2+\frac{1}{2}y^2-z^2-\frac{3}{2}(y^2+2yz+z^2)+\frac{1}{2}(x^2+2zx+z^2)-1=0$

Put (2) in this to get

$\frac{7}{2}x^2+\frac{1}{2}y^2-z^2+\frac{3}{2}($

$\Rightarrow$ From ①,

$$\frac{9}{2}x^2 + \frac{7}{2}y^2 + 3z^2 - \frac{3}{2}(y^2 - 2yz + z^2) - \frac{1}{2}(x^2 - 2xz + z^2) - 1 = 0$$

put ② in this to get

$$\frac{9}{2}x^2 + \frac{7}{2}y^2 + 3z^2 - \frac{3}{2}\cancel{(-x)^2} - \frac{1}{2}(-y)^2 - 1 = 0$$

$$\Rightarrow 3x^2 + 3y^2 + 3z^2 = 1 \quad —③$$

which is a sphere & also passes through the intersection of ① & ②

i.e. Intersection of ① & ②

= Intersection of ② & ③

i.e. intersection of a plane $x + y - z = 0$ &

Sphere $x^2 + y^2 + z^2 = \frac{1}{3}$ which is a circle

Centre of sphere is $(0,0,0)$ which also lies on the plane $x + y - z = 0$.

$\Rightarrow$ The circle required is a great circle of sphere ③ of which the radius $= \frac{1}{\sqrt{3}}$

$\Rightarrow$ Intersection of the given conicoid & plane is a circle, whose radius $= \frac{1}{\sqrt{3}}$.

SECTION-B

5. (a) Prove that $1/(x+y+1)^4$ is an integrating factor of $(2xy-y^2-y)dx + (2xy-x^2-x)dy = 0$ ①, and find the solution of this equation. (10)

Consider $M = \dfrac{2xy - y^2 - y}{(x+y+1)^4}$ & $N = \dfrac{2xy - x^2 - x}{(x+y+1)^4}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{2x-2y-1}{(x+y+1)^4} - \frac{4(x+y+1)^3 \cdot (2xy-y^2-y)}{(x+y+1)^8}$$

$$= \frac{(x+y+1)(2x-2y-1) - 4(2xy-y^2-y)}{(x+y+1)^5}$$

$$= \frac{2x^2+2y^2 - 8xy + x + y - 1}{(x+y+1)^5}$$

$$\frac{\partial N}{\partial x} = \frac{2y-2x-1}{(x+y+1)^4} - \frac{4(x+y+1)^3 \cdot (2xy-x^2-x)}{(x+y+1)^8}$$

$$= \frac{(x+y+1)(2y-2x-1) - 4(2xy-x^2-x)}{(x+y+1)^4}$$

$$= \frac{2x^2+2y^2-8xy+x+y-1}{(x+y+1)^5} = \frac{\partial M}{\partial y}$$

$\Rightarrow Mdx + Ndy = 0$ is exact

i.e. $\dfrac{1}{(x+y+1)^4}$ is an integrating factor of ①.

& solution of eqn is given by

$$\int \frac{2xy-y^2-y}{(x+y+1)^4}\,dx + \int 0\,dy = 0$$

(taking y as constant) (terms in N not containing x)

$$\text{i.e.} \int \frac{2y(x+y+1) - 3y^2 - 3y}{(x+y+1)^4}\,dx = 0$$

$$\Rightarrow \frac{-y}{(x+y+1)^2} + (y^2+y)\cdot\frac{1}{(x+y+1)^3} = C$$

$$\Rightarrow y^2+y-y(x+y+1) = C(x+y+1)^3$$

$\Rightarrow xy = C(x+y+1)^3$ where C is an arbitrary constant

5. (b) Examine for singular solution and extraneous loci, $y + px = x^4 p^2$ **(10)**

$y + px = x^4 p^2 \quad - (1)$

$\Rightarrow$ p-discriminant is

$x^2 + 4yx^4 = 0$

i.e. $x^2(1+4yx^2) = 0$

Taking $x = 0$, in (1), we get

$y = 0$, which is not necessary.

Taking $1 + 4yx^2 = 0$ in (1),

i.e. $\frac{dy}{dx} = +\frac{d}{dx}\left(-\frac{1}{4x^2}\right) = \frac{1}{2x^3}$

$\Rightarrow$ LHS of (1)

$= -\frac{1}{4x^2} + \frac{1}{2x^3} \cdot x = \frac{1}{4x^2}$

RHS of (1) $= x^4 \times \frac{1}{(2x^3)^2} = \frac{1}{4x^2} =$ LHS.

$\Rightarrow 1+4yx^2 = 0$ satisfies (1)

$\Rightarrow 1+4yx^2 = 0$ is a singular solution of (1).

For p-discriminant, we obtain ET^2C.

$\because E = 1+4yx^2$, & x is appearing twice in p-discriminant,

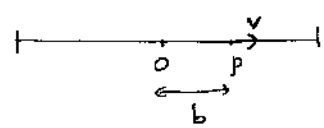
T-locus is $x = 0$

5. (c) Four uniform rods are freely jointed at their extremities and form a parallelogram $ABCD$, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the weight of all the four rods. **(10)**

5. (d) A particle is performing a simple harmonic motion of period T about centre O and it passes through a point P where $OP = b$ with velocity v in direction OP; prove that the time which elapses before it returns to P is:

$$\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$$ **(10)**

Let $a = -kx^2$



$\Rightarrow \cancel{x = \frac{1}{k}\sin(kt+\phi)}$

$\cancel{\text{At } t=0,\ x=0 \Rightarrow \phi=0}$

$\Rightarrow \cancel{x = \frac{1}{k}\sin(kt)} \ \ \cancel{v = \cos(kt)}$

5. (e) If $f = \nabla(\vec{a} \cdot \nabla r^{-1})$, show that $div\, f = 0$, and $f = curl\, g$, where $g = -\vec{a} \times \nabla(r^{-1})$. **(10)**

$$\nabla \cdot f = \nabla \cdot \left\{ \nabla \left(\bar{a} \cdot \nabla r^{-1} \right) \right\}$$

$$\nabla r^n = n r^{n-2} \bar{r}$$

$$\Rightarrow \nabla \cdot f = \nabla \cdot \left\{ \nabla \left(\bar{a} \cdot - r^{-3} \bar{r} \right) \right\}$$

$$= \nabla \cdot \left\{ \nabla \left(-\frac{\bar{a} \cdot \bar{r}}{r^3} \right) \right\} = \cancel{\nabla \cdot \{}$$

$$\cancel{= \nabla \left\{ \bar{a} \cdot \nabla \frac{\bar{r}}{r^2} \right\}}$$

$$\cancel{= \nabla \left\{ \bar{a} \left(\bar{r}\, \nabla \frac{1}{r^3} + \frac{1}{r^3} \nabla \cdot \bar{r} \right) \right\}}$$

$$= -\nabla \cdot \left\{ \bar{a} \times \left(\nabla \times \frac{\bar{r}}{r^3} \right) + \cancel{\frac{\bar{r}}{r^3}} \times (\nabla \times \bar{a}) + \left(\frac{\bar{r}}{r^3} \cdot \nabla \right) \bar{a} + (a \cdot \nabla) \frac{\bar{r}}{r^3} \right\}$$

$$= -\nabla \cdot \left\{ 0 + 0 + 0 + (a \cdot \nabla) \frac{\bar{r}}{r^3} \right\}$$

$$= 0.$$

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$$g = -\bar{a} \times \nabla(r^{-1}) = -\bar{a} \times (-r^{-3}\bar{r})$$
$$= r^{-3}(\bar{a} \times \bar{r})$$

$$\text{curl } g = \nabla \times \left(-\bar{a} \times \nabla(r^{-1})\right)$$
$$= -\left[(\nabla \cdot \nabla r^{-1} + \nabla r^{-1} \cdot \nabla)\bar{a} - (\nabla \cdot \bar{a} + \bar{a} \cdot \nabla)\nabla r^{-1}\right]$$
$$= -\left[\nabla \cdot (-r^{-3}\bar{r}) + 0 - (0 + \bar{a} \cdot \nabla)\nabla r^{-1}\right]$$
$$= 0 + \bar{a} \cdot \nabla(\nabla r^{-1}) \quad \left[\because \nabla \cdot (r^{-3}\bar{r}) = 0\right]$$
$$= \nabla(\bar{a} \cdot \nabla r^{-1})$$
$$= f.$$

6. (a) Prove that the orthogonal trajectories of the curves $A = r^2 \cos\theta$ are the curves $B = r\sin^2\theta$. **(10)**

$A = r^2 \cos\theta$ — ①

Differentiating ① by θ, we get

$$0 = 2r\cos\theta \frac{dr}{d\theta} + r^2(-\sin\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{r\sin\theta}{2\cos\theta} \quad \text{— ②}$$

To obtain orthogonal trajectories of ②

replace $\frac{dr}{d\theta}$ by $-\frac{r^2 d\theta}{dr}$

$$\Rightarrow -\frac{r^2 d\theta}{dr} = \frac{r\sin\theta}{2\cos\theta}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{-2r\cos\theta}{\sin\theta}$$

$$\Rightarrow \int \frac{dr}{r} = -2\int \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$\Rightarrow \ln r = -2 \ln \sin\theta + c_1$$

$$\Rightarrow r\sin^2\theta = e^{c_1} = B$$

where B is an arbitrary constant

i.e. The required orthogonal trajectories are given by $\underline{\underline{B = r\sin^2\theta}}$.

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6. (b) Solve the differential equation $\left(x^3D^3+3x^2D^2+xD+1\right)y = x\log x$ — (1) **(13)**

Put $x = e^z$ in (1) to get

$$\left[D_1(D_1-1)(D_1-2) + 3D_1(D_1-1) + D_1 + 1\right]y = e^z \log e^z$$

$$\left[\text{where } D_1 = \frac{d}{dz}\right]$$

$$\Rightarrow \left[D_1^3 - 3D_1^2 + 2D_1 + \cancel{(3D_1 - 3D_1) + D_1} + 1\right] y = ze^z$$

$$\Rightarrow (D_1^3 + 1)\, y = ze^z \quad \text{— (2)}$$

$\Rightarrow$ Auxiliary equation of (2) is

$$m^3 + 1 = 0$$

$$\text{i.e. } (m+1)(m^2 - m + 1) = 0$$

$\Rightarrow m = -1, \frac{1 \pm \sqrt{3}i}{2}$

$\Rightarrow CF = c_1 e^{-z} + e^{z/2}\left(c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z\right)$ — (3)

$PI = \frac{1}{D_1^3+1} z e^z$

$= e^z \cdot \frac{1}{(D_1+1)^3+1} z$

$= e^z \cdot \frac{1}{2 + 3D_1 + 3D_1^2 + D_1^3} z$

$= \frac{e^z}{2}\left[1 - \left(\frac{3D_1 + 3D_1^2 + D_1^3}{2}\right)\right] z$

$= \frac{e^z}{2}\left[z - \frac{3}{2}\right]$ — (2)

$\Rightarrow$ Solution of (1) is give by

$y = c_1 e^{-z} + e^{z/2}\left(c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z\right) + \frac{e^z}{2}\left(z - \frac{3}{2}\right)$

i.e. $y = \frac{c_1}{x} + \sqrt{x}\left(c_2 \cos\sqrt{3}\ln x + c_3 \sin\sqrt{3}\ln x\right) + \frac{x}{2}\left(\ln x - \frac{3}{2}\right)$

6. (c) Apply the method of variation of parameters to solve $y_2 + a^2 y = \cot ax$ —(1) (13)

Consider $y_2 + a^2 y = 0$ —(2)

Auxiliary equation of (2) is $m^2 + a^2 = 0$

$\Rightarrow m = \pm ai$

$\Rightarrow \cos ax$ & $\sin ax$ are the solutions of (2)

Let $y_1 = \cos ax$, $y_2 = \sin ax$ —(3)

Clearly y_1 & y_2 are linearly independent.

since $W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a\cos^2 ax + a\sin^2 ax$

$= a \neq 0.$

$\Rightarrow$ Let the solution of (1) be

$y = v_1 y_1 + v_2 y_2$ where v_1 & v_2 are functions of x.

Put this in (1) to get,

$v_1' y_1 + v_2' y_2 = 0$ & $v_1' y_1' + v_2' y_2' = \cot ax$

i.e. $v_1' \cos ax + v_2' \sin ax = 0$ — (4)

$v_1'(-a\sin ax) + v_2'(a\cos ax) = \cot ax$ —(5)

$\Rightarrow v_1 = \int -\frac{y_2 R}{W} dx = \int -\frac{\sin ax \cdot \cot ax}{a} dx$

$= \int -\frac{\cos ax}{a} dx = -\frac{\sin ax}{a^2} + C_1$

$v_2 = \int \frac{y_1 R}{W} dx = \int \frac{\cos ax \cdot \cot ax}{a} dx$

$= \int \frac{\cos^2 ax}{a \sin ax} dx = \frac{1}{a}\int \left(\frac{1}{\sin ax} - \sin ax\right) dx$

$= \frac{1}{a}\left[\frac{1}{a}\log(\operatorname{cosec} ax - \cot ax) + \frac{1}{a}\cos ax\right] + C_2$

$\Rightarrow$ Solution of ① is given by

$$y = v_1 y_1 + v_2 y_2$$

i.e. $y = C_1 \cos ax + C_2 \sin ax - \frac{1}{a^2} \cos ax \cdot \sin ax$
$+ \frac{1}{a^2} \sin ax \left[\log(\operatorname{cosec} ax - \cot ax) + \cos ax \right]$

i.e. $y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a^2} \log\left(\frac{1 - \cos ax}{\sin ax}\right)$

where C_1, C_2 are arbitrary constants.

6. (d) By using Laplace transform method solve the differential equation

$(D^2 - D - 2)y = 20 \sin 2t$, subject to initial conditions $y = -1, Dy = 2$ when $t = 0$. **(14)**

$$(D^2 - D - 2)\, y = 20 \sin 2t \quad -①$$

Applying Laplace transform on ①,

$$s^2 y(s) - s y(0) - y'(0) - [s\, y(s) - y(0)] - 2\, y(s) = 20 \cdot \left(\frac{2}{s^2 + 2^2}\right)$$

$$\Rightarrow (s^2 - s - 2)\, y(s) + s - 2 + (-1) = \frac{40}{s^2+4}$$

$$\Rightarrow (s^2 - s - 2)\, y(s) = \frac{40}{s^2+4} - (s-3)$$

$$\Rightarrow (s-2)(s+1)\, y(s) = \frac{40}{s^2+4} - (s-3)$$

$$\Rightarrow y(s) = \frac{40}{(s^2+4)(s-2)(s+1)} - \frac{(s-3)}{(s-2)(s+1)}$$

$$= \left[\frac{s-6}{s^2+4} + \frac{5/3}{s-2} + \frac{-8/3}{s+1}\right] - \left[\frac{-1/3}{s-2} + \frac{4/3}{s+1}\right]$$

$$= \frac{s-6}{s^2+4} + \frac{2}{s-2} - \frac{4}{s+1}$$

$$= \frac{s}{s^2+4} - \frac{3 \cdot 2}{s^2+4} + \frac{2}{s-2} - \frac{4}{s+1}$$

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$\Rightarrow$ Applying ~~...~~ inverse Laplace, we get

$$y(t) = \cos 2t - 3\sin 2t + 2e^{2t} - 4e^{-t}$$

8. (a) (i) If $\phi(x,y,z)=xy^2z$ and $\mathbf{f}=xz\mathbf{i}-xy\mathbf{j}+yz^2\mathbf{k}$, show that $\frac{\partial^3}{\partial x^2\partial z}(\phi\mathbf{f})$ at $(2,-1,1)$ is $\mathbf{f}=4\mathbf{i}+2\mathbf{j}$.

(ii) Prove that $\nabla\times(\mathbf{F}\times\mathbf{r})=2\mathbf{F}-(\nabla\cdot\mathbf{F})\mathbf{r}+(\mathbf{r}\cdot\nabla)\mathbf{F}$ **(12)**

(i) $\phi f = xy^2z\cdot(xzi - xyj + yz^2k)$

$= x^2y^2z^2 i - x^2y^3z j + xy^3z^3 k$ — ①

$\Rightarrow \frac{\partial}{\partial z}(\phi f) = 2x^2y^2z i - x^2y^3 j + 3xy^3z^2 k$

$\Rightarrow \frac{\partial}{\partial x}\frac{\partial}{\partial z}(\phi f) = 4xy^2z i - 2xy^3 j + 3y^3z^2 k$

$\Rightarrow \frac{\partial^3}{\partial x^2\partial z}(\phi f) = 4y^2z i - 2y^3 j + 0k$

$\therefore$ At $(2,-1,1)$, $\frac{\partial^3}{\partial x^2\partial z}(\phi f) = 4(-1)^2\cdot 1\, i - 2\cdot(-1)^3 j$

$= 4i + 2j$

(ii) $\nabla \times (F \times r)$

$$= (\nabla \cdot r + r \cdot \nabla)F - (\nabla \cdot F + F \cdot \nabla) r$$

$$\left[\because \nabla \times (A \times B) = (\nabla \cdot B + B \cdot \nabla)A - (\nabla \cdot A + A \cdot \nabla)B\right]$$

$$= (3 + r \cdot \nabla)F - (\nabla \cdot F)r - (F \cdot \nabla)r$$

$$= 3F + (r \cdot \nabla)F - (\nabla \cdot F)r - \left(F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}\right)r$$

[where $F_1 i + F_2 j + F_3 k = F$]

$$= 3F + (r \cdot \nabla)F - (\nabla \cdot F)r - (F_1 i + F_2 j + F_3 k)$$

$[\because r = xi + yj + zk]$

$$= 3F + (r \cdot \nabla)F - (\nabla \cdot F)r - F$$

$$= 2F - (\nabla \cdot F)r + (r \cdot \nabla)F$$

$$= RHS$$

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Hence proved that

$$\underline{\nabla \times (F \times r) = 2F - (\nabla \cdot F)r + (r \cdot \nabla)F}$$

8. (b) A vector function f is the product of a scalar function and the gradient of a scalar function, show that $f \cdot curl\, f = 0$ **(08)**

Let $f = \psi \nabla \phi$ where ψ & ϕ are scalar functions.

$\Rightarrow$ LHS $= f \cdot \text{curl} f$

$$= \psi \nabla \phi \cdot \nabla \times (\psi \nabla \phi)$$

$$= \psi \nabla \phi \cdot [\psi(\nabla \times \nabla \phi) + (\nabla \psi \times \nabla \phi)]$$

$[\because \nabla \times (\phi A) = \phi(\nabla \times A) + \nabla \phi \times A]$

$$= \psi \nabla \phi \cdot [0 + (\nabla \psi \times \nabla \phi)]$$

$[\because \text{curl grad} \phi = 0]$

$$\cancel{= \psi \nabla \phi \cdot (\nabla \phi \times \nabla \psi)}$$

$$= \psi \cdot \{[\nabla \psi \times \nabla \phi] \cdot \nabla \phi\} \quad [\because A \cdot B = B \cdot A]$$

$$= \psi \{\nabla \psi \cdot (\nabla \phi \times \nabla \phi)\} \quad [\because (A \times B) \cdot C = A \cdot (B \times C)]$$

$$= \psi \{\nabla \psi \cdot 0\} \quad [\because A \times A = 0]$$

$$= 0$$

$= RHS$

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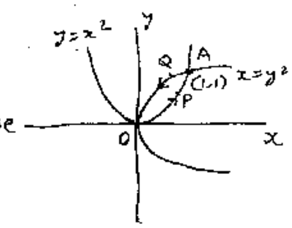
Hence, $f \cdot \text{curl} f = 0$

8. (c) Verify Green's theorem in the plane for $\int_C \left[(2xy - x^2)\,dx + (x^2 + y^2)\,dy\right]$,
where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$ described in the positive sense. **(12)**



C is the boundary represented by curves OPA & AQO is anticlockwise direction & S is the region enclosed.

Green's theorem is given by

$$\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$LHS = \oint_C \left[(2xy - x^2)\,dx + (x^2+y^2)\,dy\right]$$

$$= \int_{\substack{y=0 \\ x=y^2}}^{1} \left[(2xy - x^2)\,dx + (x^2+y^2)\,dy\right] + \int_{\substack{x=1 \\ y=x^2}}^{0} \left[(2xy - x^2)\,dx + (x^2+y^4)\,dy\right]$$

$$= \int_{y=0}^{1} (2y^3 - y^4)\cdot 2y\,dy + (y^4+y^2)\,dy + \int_{x=1}^{0} (2x^3 - x^2)\,dx + (x^2+x^4)\,2x\,dx$$

$$= \int_{y=0}^{1} (-2y^5 + 5y^4 + y^2)\,dy + \int_{x=1}^{0} (2x^5 + 4x^3 - x^2)\,dx$$

$$= \left(-\frac{1}{3}y^6 + y^5 + \frac{y^3}{3}\right)\Big|_0^1 + \left(\frac{x^6}{3} + x^4 - \frac{x^3}{3}\right)\Big|_1^0$$

$$= -\frac{1}{3} + 1 + \frac{1}{3} - \left(\frac{1}{3} + 1 - \frac{1}{3}\right) = 0.$$

$$RHS = \iint_S \left\{\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right\} dx\,dy$$

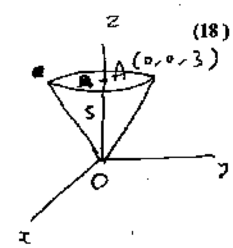
$$= \iint_S (2x - 2x)\,dx\,dy = 0$$

i.e. LHS = RHS

$\Rightarrow$ Green theorem is verified.

8\. (d) By using Gauss divergence theorem evaluate $\iint_S (x^2+y^2)\,dS$, where S is the surface of the cone $z^2=3(x^2+y^2)$ bounded by $z=0$ and $z=3$.

(18)



Volume V is enclosed by the surface of the cone $z^2 = 3(x^2+y^2)$ & $z=0$ & $z=3$.

Gauss divergence theorem is given by

$$\iint_S \bar{A}\cdot d\bar{s} = \iiint_V \nabla\cdot\bar{A}\,dv$$

Given $I = \iint_S (x^2+y^2)\,ds$

Let $f(x,y,z) = 3(x^2+y^2) - z^2$

$\Rightarrow$ normal, $\bar{n} = \nabla f = 6xi + 6yj - 2zk$

$\Rightarrow$ unit normal, $\hat{n} = \dfrac{6xi + 6yj - 2zk}{\sqrt{36x^2+36y^2+4z^2}}$

$= \dfrac{6xi + 6yj - 2zk}{\sqrt{12z^2+4z^2}}$

$= \dfrac{1}{4z}(6xi + 6yj - 2zk)$

$= \dfrac{1}{2z}(3xi + 3yj - zk)$

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$\cancel{\Rightarrow \bar{A}\cdot\hat{n}\, ds}$

$\Rightarrow \bar{A}\cdot d\bar{s} = \bar{A}\cdot\hat{n}\, ds = x^2+y^2$

$\Rightarrow \bar{A}\cdot \frac{1}{2z}(3xi + 3yj - zk) = x^2+y^2$

$\Rightarrow \bar{A}\cdot\left(\frac{3}{2}\frac{x}{z} i + \frac{3}{2}\frac{y}{z} j - \frac{1}{2}k\right) = x^2+y^2$

$\Rightarrow$ Take $\bar{A} \to -2(x^2+y^2)k$

$\Rightarrow$ By Gauss divergence theorem

$I = \iint_S -2(x^2+y^2)k \cdot d\bar{s} \; \cancel{= \iiint_V -2(x^2+y^2)k}$

$= \iiint_V \nabla\cdot\{-2(x^2+y^2)k\}\, dv$

$= \iiint_V -2\, \nabla\cdot\left[(x^2+y^2)k\right] dv = 0.$