

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-I(M) IAS / T-07

MATHEMATICS

by K. VENKANNA

The person with 14 years of Teaching Experience

FULL TEST P-I



Time: Three Hours

Maximum Marks: 250

INSTRUCTIONS

- This question paper-cum-answer booklet has 50 pages and has 33 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
- Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
- A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
- Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
- Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
- The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
- Symbolnotations carry their usual meanings, unless otherwise indicated.
- All questions carry equal marks.
- All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
- All rough work should be done in the space provided and scored out finally.
- The candidate should respect the instructions given by the invigilator.
- The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name

Vanun Guntupalli

Roll No.

Test Centre Hyderabad

Medium English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Evar

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted coniguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

**DO NOT WRITE ON
THIS SPACE**

P.T.O.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
	(c)			
	(d)			23
	(e)			
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			
	(b)			
	(c)			19
	(d)			12
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			
	(b)			
	(c)			
	(d)			
	(e)			24
6	(a)			
	(b)			
	(c)			
	(d)			22
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			41
Total Marks				

122

PTO

**DO NOT WRITE ON
THIS SPACE**



Head Office: 105-106, Top Floor, Mukherjee Tower, Dr. M. Mukherjee Nagar, Delhi-110009.
Branch Office: 25A, Old Rajender Nagar Market, Dehradoon-248009
Ph. 011-48629887, 09899329111, 91988197829 | www.imsdelhi.com | www.imsdelhilearning.com | imsdelhi2012@gmail.com

P.T.O.

SECTION-A

1. (a) (i) If $A^2 = I$, what are the possible eigen values of A ?
(ii) If this A is 2 by 2, and not I or $-I$, find its trace and determinant.
(iii) If the first row is $(3, -1)$, what is the second row.

(i) Given $A^2 - I = 0$

(10)

\Rightarrow We know that a square matrix A is the root of its own characteristic equation by Cauchy Hamilton theorem.

\Rightarrow characteristic equation of A is

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

\Rightarrow Possible eigen values of $A = 1, -1$

(ii) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^2 = I \Rightarrow \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow b(a+d) = c(a+d) = 0 ; a^2 + bc = d^2 + bc = 1$$

$$\therefore A \neq I \neq -I, b \neq 0 \Rightarrow b + c \neq 0$$

Hence $a = -d$ & $a^2 + bc = 1$

Or \Rightarrow Trace = $a + d = 0$

$$\text{Determinant} = ad - bc = -a^2 - bc = -1$$

(iii) First row is $(3, -1) \Rightarrow a = 3$ & $b = -1$

$$\Rightarrow d = -3 \text{ & } c = \frac{1-a^2}{b} = \frac{1-9}{-1} = 8$$

\Rightarrow 2nd row is $(8, -3)$

1. (b) Obtain eigen values and eigen vectors of the differential operator $D : P_2 \rightarrow P_2$
 $D(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$, for $a_0, a_1, a_2 \in \mathbb{R}$.

(10)

$$(a_0, a_1, a_2) \cdot D \cdot \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = (a_1, 2a_2, 0) \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

$$\Rightarrow D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \quad \boxed{D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}$$

\Rightarrow characteristic eqn of D is $|\lambda D - I| = 0$

$$\Rightarrow \begin{vmatrix} \lambda & 0 & 0 \\ -1 & \lambda & 0 \\ 0 & -2 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 = 0$$

$\Rightarrow \lambda = 0, 0, 0$

\Rightarrow Eigen values = 0, 0, 0.

$$D\bar{x} = \lambda \bar{x}$$

For $\lambda = 0$,

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow x_2 = 0, x_1 = 0.$$

\Rightarrow Eigen vector = $(0, 0, 1)$.
 No more again

$$(1, 0, 0)$$

1. (c) Let ϕ be a function of two variables defined as

$$\begin{aligned}\phi(x, y) &= (x^2 + y^2)/(x - y), && \text{when } x \neq y \\ \phi(x, y) &= 0, && \text{when } x = y\end{aligned}$$

Show that ϕ is discontinuous at the origin, but the first order partial derivatives exist at that point. (10)

$$\begin{aligned}\text{At } (0, 0), \frac{\partial \phi}{\partial x} &= \lim_{h \rightarrow 0} \frac{\phi(h, 0) - \phi(0, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 - 0}{h} = 0.\end{aligned}$$

$$\begin{aligned}\text{At } (0, 0), \frac{\partial \phi}{\partial y} &= \lim_{k \rightarrow 0} \frac{\phi(0, k) - \phi(0, 0)}{k} \\ &= \lim_{k \rightarrow 0} \frac{-k^2 - 0}{k} = 0.\end{aligned}$$

\Rightarrow 1st order partial derivatives exist $\Rightarrow \phi(x, y)$ at the origin.

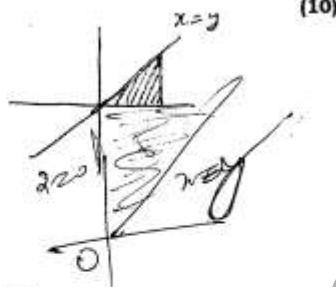
1. (d) Let $E = \{(x, y) \in \mathbb{R}^2 / 0 < x < y\}$. Then evaluate

$$\iint_E ye^{-(x+y)} dx dy$$

$$\begin{aligned} I &= \int_0^\infty \int_{y=0}^\infty ye^{-x-y} e^{-y} dx dy \\ &= \int_0^\infty ye^{-2y} (e^{-x}) \Big|_0^\infty dy \end{aligned}$$

O2

$$\begin{aligned} &= \int_0^\infty ye^{-2y} (0 + e^{-2y}) dy \\ &= \int_0^\infty ye^{-4y} dy \\ &= \left[y \cdot -\frac{1}{2} e^{-4y} \right]_0^\infty - \int_0^\infty -\frac{1}{2} e^{-4y} dy \\ \text{cheer up again} &= 0 + \frac{1}{2} \times -\frac{1}{2} e^{-4y} \Big|_0^\infty \\ &= -\frac{1}{4} (0 - 1) \\ &= \boxed{\frac{1}{4}} \end{aligned}$$



(10)

1. (e) Find the distance of the point $(1, -2, 3)$ from the plane $x - y + z = 5$ measured parallel to the line $\frac{1}{2}x = \frac{1}{3}y = -\frac{1}{6}z$.

Let the line from $P(1, -2, 3)$

in the direction of line $\frac{x}{2} = \frac{y}{3} = -\frac{z}{6}$

intersect plane $x - y + z = 5$ in A

$$\Rightarrow \text{Line PA is } \frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = \lambda \quad (1)$$

$\Rightarrow (2\lambda+1, 3\lambda-2, -6\lambda+3)$ lies on PA

$\Rightarrow A$ is given by

$$2\lambda+1 - (3\lambda-2) + (-6\lambda+3) = 5$$

$$\Rightarrow -7\lambda + 6 = 5 \Rightarrow \lambda = 1/7$$

$$\Rightarrow A \text{ is } \left(\frac{2}{7}+1, \frac{3}{7}-2, -\frac{6}{7}+3\right)$$

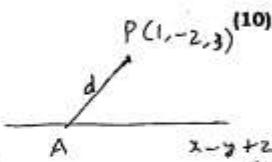
$$\text{i.e. } \left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$$

$$\Rightarrow d^2 = AP^2 = \left(\frac{9}{7}-1\right)^2 + \left(\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2$$

$$= \frac{4+9+36}{49} = 1$$

$$\Rightarrow AP = 1$$

i.e. distance of P from the plane $x - y + z = 5$
measured " to the given line = 1



2. (a) (i) If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, find A^{100} by diagonalizing A .

(ii) Show by direct calculation that AB and BA have the same trace when $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

and $B = \begin{bmatrix} q & r \\ s & t \end{bmatrix}$. Deduce that $AB - BA = I$ is impossible (except in infinite dimensions).

(10+5=15)

IMS-Institute Of Mathematical Sciences



Head Office: 105-106, Top Floor, Muthuraja Tower, Dr. Muthuraja Nagar, Delhi-110066.
Branch Office: 283, Old Rajender Nagar Market, De-84-10066
Ph: 011-49429987, 9999929111, 09999197525 | www.imsdelhi.com | www.ims4mathstutoring.com | Emailims4maths@gmail.com

P.T.O.

2. (d) Let V and W be subspaces of \mathbb{R}^3 defined as follows:
 $V = \{(a, b, c) / b + 2c = 0\}$, $W = \{(a, b, c) / a + b + c = 0\}$
Find bases and dimensions of V , W , $V \cap W$. Hence prove that

$$V + W = \mathbb{R}^3$$

(10)

3. (a) Discuss the nature of the critical values of $f(x, y) = e^x \sin y$. (10)

Critical values of $f(x, y)$ are given by

$$\frac{\partial f}{\partial x} = 0 \text{ & } \frac{\partial f}{\partial y} = 0$$

i.e. $e^x \sin y = 0$ & $e^x \cos y = 0$

3. (b) Let $z = f(t)$, $t = \frac{x+y}{xy}$. Show that $x \frac{\partial z}{\partial x} = y \frac{\partial z}{\partial y}$ (10)

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial x} = t'(t) \cdot \frac{\partial}{\partial x} \left(\frac{1}{x} + \frac{1}{y} \right)$$

$$= t'(t) \cdot \left[-\frac{1}{x^2} \right] \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial y} = t'(t) \cdot \frac{\partial}{\partial y} \left(\frac{1}{x} + \frac{1}{y} \right)$$

$$= t'(t) \cdot \left[-\frac{1}{y^2} \right] \quad \text{--- (2)}$$

$$\Rightarrow LHS = x^2 \frac{\partial z}{\partial x} = x^2 \cdot t'(t) \cdot \left[-\frac{1}{x^2} \right] = -t'(t)$$

$$= y^2 \cdot t'(t) \cdot \left[-\frac{1}{y^2} \right] = y^2 \frac{\partial z}{\partial y} = RHS$$

Q8. Hence, $\underline{x^2 \frac{\partial z}{\partial x}} = \underline{y^2 \frac{\partial z}{\partial y}}$.

3. (c) Find the maximum and minimum values of $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disc $x^2 + y^2 \leq 1$ (15)

For max & min. values of $f(x, y)$,

$$\frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 2x = 0 \quad \Rightarrow 6y + 2 = 0$$

Let $g(x, y) = x^2 + 3y^2 + 2y + \lambda(x^2 + y^2 - 1)$

use $\lambda \in [0, 1]$

$$\frac{\partial g}{\partial x} = 0 \Rightarrow 2x + 2x\lambda = 0 \quad \text{---(1)}$$

$$\frac{\partial g}{\partial y} = 0 \Rightarrow 6y + 2 + 2y\lambda = 0 \quad \text{---(2)}$$

$$\Rightarrow x^2 + y^2 = a \quad \text{---(3)}$$

$$\Rightarrow x = 0 \text{ & } \lambda = -1 \text{ from } ①$$

$$\text{If } x=0, y = \pm\sqrt{a} - ④$$

$$\text{If } \lambda = -1, y = -\frac{1}{2}, x^2 + \frac{1}{4} = a^2$$

$$\Rightarrow \text{If } y = -\frac{1}{2}, x = \pm\sqrt{a - \frac{1}{4}} - ⑤$$

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 6$$

$$\Rightarrow AC - B^2 = 12 > 0 \text{ & } x, y$$

$$\text{& } A > 0$$

$\Rightarrow f(x, y)$ is minimum at the critical values.

3. (d) Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$ bounded by the planes $z = x \tan \alpha$ and $z = x \tan \beta$ where $\beta > \alpha$. (15)

IMS Institute Of Mathematical Sciences

IMS™

INSTITUTE OF MATHEMATICAL SCIENCES

Head Office: 105-116, Top Floor, Mukherjee Tower, Dr. Mukherjee Nagar, Delhi-110091.

Branch Office: 25A, Old Rajender Nagar Market, Delhi-110060.

Ph. 011-45829567, 69999320111, 09999157623 | www.imsindia.com | www.imsindialearning.com | Email:ims2016@gmail.com

P.T.O.

IMS-Institute Of Mathematical Sciences



Head Office: 104-106, Two Floor, Mathura Road, Dr. Nehru Nagar, Delhi-110091.
Branch Office: 104, Old Railway Nagar Market, Delhi-110090
Ph: 911-45629987, 09999326111, 09999197922 E: www.imsinstitute.com E: imsinstitute2010@gmail.com

SECTION-B

5. (a) Find the orthogonal trajectories of cardioids $r = \alpha(1 - \cos \theta)$, α being parameter. (10)

$$\frac{dr}{d\theta} = +\alpha \sin \theta \quad \text{(1)}$$

\Rightarrow Differential eqn of given equation is

$$r = \frac{1}{\sin \theta} \frac{dr}{d\theta} (1 - \cos \theta) \quad \text{(2)}$$

Put $\frac{dr}{d\theta} = -\frac{r^2 d\theta}{2r}$ in place of $\frac{dr}{d\theta}$ for orthogonal trajectories.

$$\Rightarrow r \sin \theta = -\frac{r^2 d\theta}{2r} (1 - \cos \theta)$$

$$\Rightarrow -\frac{d\theta}{r} = \frac{d\theta (1 - \cos \theta)}{\sin \theta}$$

$$\Rightarrow -\frac{d\theta}{r} = \frac{2 \sin^2 \theta/2}{2 \sin \theta/2 \cos \theta/2} d\theta = \tan \theta/2 d\theta$$

$$\Rightarrow -\ln r = -\ln \cos(\theta/2) + C_1$$

$$\Rightarrow -\ln r = -2 \ln \cos \theta/2 + C_1$$

$$\Rightarrow r = C_2 (\cos^2 \theta/2)$$

$$\Rightarrow r = C (1 + \cos \theta) \text{ where } C = C_2/2$$

\therefore Orthogonal trajectories of given eqn is

$$r = C(1 + \cos \theta) \text{ where } C \text{ is an arbitrary constant.}$$

5. (b) Solve $(D^2+1)^2y = 24x \cos x$ given the initial conditions $x=0, y=0, Dy=0, D^2y=0, D^3y=12$. (10)

Auxiliary eqn is $(m^2+1)^2 = 0$
 $\Rightarrow m = \pm i, \pm i$

~~$\Rightarrow Cf = (Ax+B)$~~

$$\Rightarrow Cf = (c_1 x + c_2) \cos x + (c_3 x + c_4) \sin x \quad \text{--- (1)}$$

$$PI = \frac{1}{(D^2+1)^2} 24x \cos x$$

$$\begin{aligned} \cancel{PI} &= 24x \cdot \frac{1}{(D^2+1)^2} \cos x - \frac{24 \frac{d}{dD} (D^2+1)^2}{(D^2+1)^4} \cos x \\ &= x \frac{1}{V(D)} \times V \\ &= x \frac{1}{V(D)} V - \frac{b'(D)}{(V(D))^2} V \end{aligned}$$

~~$\Rightarrow PI = 24x \cdot \frac{1}{(D^2+1)^2} \cos x - \frac{24 \frac{d}{dD} (D^2+1)^2}{(D^2+1)^4} \cos x$~~

5. (c) A particle whose mass is m is acted upon by a force $m\mu \left[x + \frac{a^4}{x^3} \right]$ towards origin. If it starts from rest at a distance a , show that it will arrive at origin in time $\pi/(4\sqrt{\mu})$. (10)

$$\text{At } x = a, \frac{dx}{dt} = 0, t = 0$$

$$a = \frac{d^2x}{dt^2} = \mu \left[x + \frac{a^4}{x^3} \right] = v \frac{dv}{dx}$$



~~$$\frac{d^2x}{dt^2} = \mu \left[\frac{x^2}{2} - \frac{1}{2} \frac{a^4}{x^2} \right] + C_1$$~~

~~$$\text{At } x = a, \frac{dx}{dt} = 0 \Rightarrow C_1 = -\mu \left[\frac{a^2}{2} - \frac{1}{2} \frac{a^4}{a^2} \right] = 0$$~~

~~$$\Rightarrow \frac{dx}{dt} = \mu \left[\frac{x^2}{2} - \frac{1}{2} \frac{a^4}{x^2} \right]$$~~

~~$$\Rightarrow x = \mu \left[\frac{x^3}{6} + \frac{1}{2} \cdot \frac{a^4}{x} \right]$$~~

$$\Rightarrow \frac{1}{2} dv^2 = -\mu \left[x + \frac{a^4}{x^3} \right] dx$$

$$\Rightarrow v^2 = -2\mu \left[x^2 - \frac{a^4}{x^2} \right] + C_1$$

$$\text{At } x=a, v=0 \Rightarrow 0 = -\mu \left[a^2 - a^2 \right] + C_1 \\ \Rightarrow C_1 = 0.$$

$$\Rightarrow \left(\frac{dx}{dt} \right)^2 = -\mu \left[x + \frac{a^2}{x} \right] \left[x - \frac{a^2}{x} \right]$$

$$\Rightarrow dt = \frac{1}{\mu} \frac{dx}{\sqrt{-x^2 + a^4/x^2}} = \frac{1}{\mu} \frac{x dx}{\sqrt{-x^4 + a^4}}$$

$$\text{put } x^2 = t$$

$$\Rightarrow dt = \frac{1}{\sqrt{\mu}} \cdot \frac{1}{2} \frac{dt}{\sqrt{t^2 + a^4}} \Rightarrow t = \frac{1}{2\sqrt{\mu}} \sin^{-1} \frac{x^2}{a^2} + C_2$$

$$\text{At } x=a, t=a \Rightarrow 0 = \frac{1}{2\sqrt{\mu}} \sin^{-1} \frac{a^2}{a^2} + C_2$$

$$\Rightarrow C_2 = \frac{\pi}{4\sqrt{\mu}}$$

$$\Rightarrow t = \frac{\pi}{4\sqrt{\mu}} + \frac{1}{2\sqrt{\mu}} \sin^{-1} \frac{x^2}{a^2} \Rightarrow \text{At } x=0, t = \frac{\pi}{4\sqrt{\mu}}$$

5. (d) A solid frustum of a paraboloid of revolution of height h and latus rectum $4a$, rests with its vertex on the vertex of a paraboloid of revolution, whose latus rectum is $4b$, show that the

equilibrium is stable if $h < \frac{3ab}{a+b}$

(10)

5. (e) Find the work done in moving a particle once around a circle C in the xy -plane, if the circle has centre at the origin and radius 2 and if the force field F is given by

$$F = (2x - y + 2z)\mathbf{i} + (x + y - z)\mathbf{j} + (3x - 2y - 5z)\mathbf{k}. \quad (10)$$

$$W = \oint_C F \cdot d\mathbf{r} \quad \text{where } C \text{ is circle in } xy\text{-plane} \\ x^2 + y^2 = 4, z = 0$$

$$\begin{aligned} W &= \oint_C [(2x - y)\mathbf{i} + (x + y)\mathbf{j} + (3x - 2y)\mathbf{k}] \cdot (dx\mathbf{i} + dy\mathbf{j}) \\ &= \oint_C [(2x - y)dx + (x + y)dy] \\ &\quad \text{Put } x = 2\cos\theta, y = 2\sin\theta \\ &= \int_0^{2\pi} [(2 + 2\cos\theta - 2\sin\theta)(-2\sin\theta d\theta) + (2\cos\theta + 2\sin\theta)(2\cos\theta d\theta)] \\ &= \int_0^{2\pi} (-4\sin\theta\cos\theta + 4)d\theta \end{aligned}$$

$$\text{circled } \int = 2 \cos 2\theta \Big|_0^{2\pi} + 4\theta \Big|_0^{2\pi}$$

$$\text{circled } \int = 8\pi$$

6. (a) Solve $(1 - x^2 y^2) dx = y dx + x dy$.

(05)

~~$$(x^2 y^2 + 1 - 1) dx + x dy = 0$$~~
~~(1)~~

~~$$\text{This is in the form } M dx + N dy = 0$$~~

~~$$\Rightarrow \frac{dy}{dx} + \frac{1 - x^2 y^2}{x} = \frac{1}{x}$$~~

~~$$\textcircled{1} \Rightarrow (1 - x^2 y^2) dx = d(xy)$$~~

~~$$\Rightarrow dx = \frac{d(xy)}{1 - (xy)^2}$$~~

~~$$\Rightarrow dx = \frac{1}{2} \left(\frac{1}{1 - xy} + \frac{1}{1 + xy} \right) d(xy)$$~~

~~$$\Rightarrow x = \frac{1}{2} \left[\ln(1 + xy) - \ln(1 - xy) \right] + C_1$$~~

~~$$\text{O/P} \Rightarrow \frac{1 + xy}{1 - xy} = C e^{2x}$$~~

where C is an arbitrary constant.

6. (b) Solve the differential equation $(px^2 + y^2)(px + y) = (p + 1)^2$ by reducing it to Clairaut's form and find its singular solution. **(15)**

IMS-Institute Of Mathematical Sciences



Head Office: 185-186, Top Floor, Mukherjee Tower, Dr. Matherjee Marg, Delhi-110096.
Branch Office: 258, C-Block Rani Durgavati Market, Delhi-110089
Ph. 011-41620887, 09899229711, 0989977623 | www.imsresults.com | www.centraltestbanking.com | imsclassmate2010@gmail.com

P.T.O.

6. (c) Solve $(x+2)y'' - (2x+5)y' + 2y = (x+1)e^x$ — (1)

(15)

$u = e^{2x}$ is a solution of

$$(x+2)y'' - (2x+5)y' + 2y = 0$$

$$[(x+2) \cdot (4e^{2x}) - (2x+5)(2e^{2x}) + 2e^{2x} = 0]$$

Re-writing (1) as $y'' - \left(\frac{2x+5}{x+2}\right)y' + \frac{2}{x+2}y = \frac{x+1}{x+2}e^x$

is of the form $y'' + P y' + Qy = R$

05 We know that

$$v_2 + \left(P + \frac{2u_1}{u}\right)v_1 = \frac{R}{u} \text{ where } u \text{ is a soln of L.H.S. eqn.}$$

$$\Rightarrow v_2 + \left(-\frac{2x+5}{x+2} + 4\right)v_1 = \frac{x+1}{x+2} \cdot e^{-x}$$

$$\Rightarrow v_2 + \left(\frac{-1}{x+2} \right) v_1 = \frac{x+1}{x+2} e^{-x}$$

$$IF = e^{\int \frac{1}{x+2} dx} = e^{\ln(x+2)} = \frac{1}{x+2}$$

$$\Rightarrow v_1 \cdot \frac{1}{x+2} = \int \frac{x+1}{(x+2)^2} e^{-x} dx$$

$$= \int \left(\frac{e^{-x}}{x+2} - \frac{e^{-x}}{(x+2)^2} \right) dx$$



$$= \frac{1}{x+2} \cdot (-e^{-x}) - \int \frac{-1}{(x+2)^2} \cdot (-e^{-x}) dx$$

6. (d) By using Laplace transform, solve $(D+9)y = \cos 2t$ with $y(0) = 1$, $y(\pi/2) = -1$. (15)

$$(D+9)y = \cos 2t \quad \text{--- (1)}$$

Applying Laplace transform,

$$sY(s) - y(0) + 9Y(s) = \frac{s}{s^2 + 4}$$

$$\Rightarrow (s+9)Y(s) - 1 = \frac{s}{s^2 + 4}$$

$$\Rightarrow Y(s) = \frac{s}{(s+9)(s^2+4)} + \frac{1}{s+9}$$

$$\frac{-9}{s+9} + \frac{9}{s^2+4} + \frac{1}{s+9}$$

$$\begin{aligned} -\beta' &= \frac{-9/85}{s+9} + \frac{\frac{9}{85}s^2 + \frac{4}{85}}{s^2+4} + \frac{1}{s+9} \\ &= \frac{1}{85} \left[\frac{76}{s+9} + \frac{9s^2 + 4}{s^2+4} \right] \end{aligned}$$

$$Y(s) = \frac{1}{85} \left[\frac{76}{s+9} + \frac{9s+4}{s^2+4} \right]$$

$$\Rightarrow y(t) = \frac{76}{85} e^{-9t} + \frac{9}{85} \cos 2t + \frac{2}{85} \sin 2t$$

8. (a) A particle moves along the curve $x = e^t$, $y = 2 \cos 3t$, $z = 2 \sin 3t$. Determine the velocity and acceleration at any time t and their magnitudes at $t = 0$. (08)

$$\vec{s} = e^{-t} i + 2 \cos 3t j + 2 \sin 3t k$$

$$\Rightarrow \vec{v} = \frac{d\vec{s}}{dt} = -e^{-t} i - 6 \sin 3t j + 6 \cos 3t k$$

a) Velocity at any time t

$$= \underline{-e^{-t} i - 6 \sin 3t j + 6 \cos 3t k}$$

Magnitude of \vec{v} at $t = 0$

$$06 = \sqrt{(-1)^2 + 0 + 6^2} = \underline{\sqrt{37}}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \underline{e^{-t} i - 18 \cos 3t j - 18 \sin 3t k}$$

$$\text{Magnitude of } \vec{a} \text{ at } t = 0 \text{ is } \sqrt{(11^2 + (18)^2 + 0)} = \sqrt{325} = 5\sqrt{13}$$

8. (b) (i) Prove that $\operatorname{div} \operatorname{grad} r^n = n(n+1) r^{n-2}$,

$$\nabla^2 r^n = n(n+1) r^{n-2}.$$

(ii) If r is the position vector of the point (x, y, z) , show that $\operatorname{curl} \{r^n \mathbf{r}\} = 0$, where r is the module of r . (6+4=10)

$$(i) \text{ LHS} = \nabla \cdot (\nabla x^n)$$

$$= \nabla \cdot [n x^{n-2} \bar{x}]$$

$$= n \cdot [x^{n-2} \nabla \cdot \bar{x} + \nabla x^{n-2} \cdot \bar{x}]$$

$$[\because \nabla(\phi \bar{A}) = \phi(\nabla \cdot \bar{A}) + \nabla \phi \cdot \bar{A}]$$

$$= n \cdot [3x^{n-2} + \bar{x} \cdot (n-2)x^{n-4} \bar{x}]$$

$$[\because \nabla x^n = n x^{n-2} \bar{x}]$$

$$= n [3x^{n-2} + (n-2)x^{n-2}] = n(n+1)x^{n-2} = \text{RHS.}$$

$$\begin{aligned}
 LHS &= \nabla^2 \lambda^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \lambda^n \\
 &= \sum \frac{\partial}{\partial x} \left(n \cdot \lambda^{n-1} \cdot \frac{\partial \lambda}{\partial x} \right) \\
 &= \sum \frac{\partial}{\partial x} \left(n \cdot \lambda^{n-1} \cdot \frac{x}{\lambda} \right) \\
 &= \sum \frac{\partial}{\partial x} \left(n \cdot \lambda^{n-2} \cdot x \right) \\
 &= n \cdot \sum \left(\frac{\partial x}{\partial x} \cdot \lambda^{n-2} + x \cdot (n-2) \lambda^{n-3} \cdot \frac{\partial \lambda}{\partial x} \right) \\
 &= n \left[3 \lambda^{n-2} + \sum (n-2) \lambda^{n-3} \cdot \frac{x^2}{\lambda} \right] \\
 &= n \left[3 \lambda^{n-2} + (n-2) \lambda^{n-4} \cdot (x^2 + y^2 + z^2) \right] \\
 &= n \left[3 \lambda^{n-2} + (n-2) \lambda^{n-2} \right] \\
 &= n(n+1) \lambda^{n-2} = RHS
 \end{aligned}$$

<08

$$\begin{aligned}
 (ii) \quad \nabla \times (\lambda^n \bar{a}) &= \nabla \lambda^n \times \bar{a} + \lambda^n (\nabla \times \bar{a}) \\
 &\quad \left[\because \nabla \times (\phi A) = \nabla \phi \times A + \phi (\nabla \times A) \right] \\
 &= n \lambda^{n-2} \bar{a} \times \bar{a} + \lambda^n \times 0 \\
 &= n \lambda^{n-2} \times 0 + 0 \\
 &= 0 = RHS
 \end{aligned}$$

8. (c) Find the curvature and torsion of the circular helix $x = a \cos \theta$, $y = a \sin \theta$, $z = a\theta \cot \alpha$

(12)

$$\vec{r}(\theta) = a \cos \theta \mathbf{i} + a \sin \theta \mathbf{j} + a\theta \cot \alpha \mathbf{k}$$

We know that

$$\text{curvature } k = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \quad \& \quad \text{Torsion } \tau = \frac{[\dot{\vec{r}} \quad \ddot{\vec{r}} \quad \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

$$\dot{\vec{r}}(\theta) = -a \sin \theta \mathbf{i} + a \cos \theta \mathbf{j} + a \cot \alpha \mathbf{k} \quad \text{--- (1)}$$

$$\ddot{\vec{r}}(\theta) = -a \cos \theta \mathbf{i} - a \sin \theta \mathbf{j} \quad \text{--- (2)}$$

$$\dddot{\vec{r}}(\theta) = a \sin \theta \mathbf{i} - a \cos \theta \mathbf{j} \quad \text{--- (3)}$$

$$\begin{aligned}\dot{\vec{r}} \times \ddot{\vec{r}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin \theta & a \cos \theta & a \cot \alpha \\ -a \cos \theta & -a \sin \theta & 0 \end{vmatrix} \\ &= \mathbf{i} (a^2 \sin \theta \cot \alpha) + \mathbf{j} (-a^2 \cos \theta \cot \alpha) \\ &\quad + \mathbf{k} (a^2 \sin^2 \theta + a^2 \cos^2 \theta) \\ &= a^2 \sin \theta \cot \alpha \mathbf{i} - a^2 \cos \theta \cot \alpha \mathbf{j} + a^2 \mathbf{k} \quad \text{--- (4)}\end{aligned}$$

$$\begin{aligned}[\dot{\vec{r}} \quad \ddot{\vec{r}} \quad \dddot{\vec{r}}] &= (\dot{\vec{r}} \times \ddot{\vec{r}}) \cdot \ddot{\vec{r}} \\ &= a^3 \sin^2 \theta \cot \alpha + a^3 \cos^2 \theta \cot \alpha \\ &= a^3 \cot \alpha \quad \text{--- (5)}\end{aligned}$$

$$\begin{aligned}|\dot{\vec{r}} \times \ddot{\vec{r}}| &= \sqrt{a^4 \cot^2 \alpha (\sin^2 \theta + \cos^2 \theta) + a^4} = a^2 \sqrt{\cot^2 \alpha + 1} \\ &= a^2 \cosec \alpha \quad \text{--- (6)}\end{aligned}$$

$$|\dot{\vec{r}}| = \sqrt{a^2 \sin^2 \theta + a^2 \cos^2 \theta + a^2 \cot^2 \alpha} = a \cosec \alpha$$

$$\Rightarrow k = \frac{a^2 \cot \alpha}{(a \cot \alpha)^3} = \frac{1}{a \cot^2 \alpha}$$

$$\tau = \frac{a^3 \cot \alpha}{(a^2 \cot^2 \alpha)^{1/2}} = \frac{a \cot \alpha \sin^2 \alpha}{\sqrt{a^2 \cot^2 \alpha}} = \frac{\cot \alpha \sin^2 \alpha}{\sqrt{a}}$$

8. (d) If $F = (y^2 + z^2 - x^2)\mathbf{i} + (z^2 + x^2 - y^2)\mathbf{j} + (x^2 + y^2 - z^2)\mathbf{k}$, evaluate $\iint_S \operatorname{curl} F \cdot dS$ taken over the portion of the surface $x^2 + y^2 + z^2 - 2ax + az = 0$ above the plane $z = 0$, and verify Stoke's theorem. (20)

Stoke's theorem is given by

$$\iint_S (\nabla \times F) \cdot dS = \oint_C F \cdot d\mathbf{r}$$

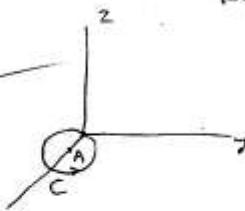
S is $x^2 + y^2 + z^2 - 2ax + az = 0$ above $z=0$.

Let $S_1 = S + C$, where S_1 is area enclosed by C .

C is $x^2 + y^2 - 2ax = 0, z = 0$

which is a circle with centre $(a, 0, 0)$

$$\begin{aligned} \nabla \times F &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 - x^2 & z^2 + x^2 - y^2 & x^2 + y^2 - z^2 \end{vmatrix} \\ &= i(2y - 2z) + j(2z - 2x) + k(2x - 2y) \end{aligned}$$



$$= 2(y-z)i + 2(z-x)j + 2(x-y)k$$

S is $\int = x^2 + y^2 + z^2 - 2xz + az = 0$

$$\Rightarrow \hat{n} = \frac{\nabla F}{|\nabla F|} = (2x-2a)i + 2yj + (2z+a)k$$

$$\Rightarrow \text{unit normal } \hat{n} = \frac{\nabla F}{|\nabla F|}$$

$$= \frac{2(x-a)i + 2yj + (2z+a)k}{\sqrt{4(x-a)^2 + 4y^2 + (2z+a)^2}}$$

$$= \frac{2(x-a)i + 2yj + (2z+a)k}{\sqrt{4(x^2 + y^2 + z^2 - 2xz + az) + 4a^2 + a^2}}$$

$$= \frac{1}{\sqrt{5}a} [2(x-a)i + 2yj + (2z+a)k]$$

$$\Rightarrow LHS = \iint_S (\nabla \times F) \cdot \hat{n} ds = \iint_S \frac{1}{\sqrt{5}a} [4(x-a)(y-z) + 4y(z-x) + 2(2z+a)(x-y)] ds$$

$$= \iint_S \frac{1}{\sqrt{5}a} [4(xy - ay - yz + az) + 4yz - 4xy + 2(2yz - 2yz - ay + ax)] ds$$

$$= \iint_S \frac{1}{\sqrt{5}a} [-6ay + 4az + 2ax] ds$$

$$LHS = \iint_S (\nabla \times F) \cdot \hat{n} ds = \iint_{S'} (\nabla \times F) \cdot \hat{n} ds - \iint_{S_1} (\nabla \times F) \cdot \hat{n} ds$$

$$= 0 - \iint_{S_1} (\nabla \times F) \cdot \hat{n} ds \quad [\text{By Gauss divergence theorem}]$$

$$= - \iint_{S_1} 2[(y-z)i + (z-x)j + (x-y)k] \cdot (-k) ds$$

$$= \iint_{S_1} 2(x-y) ds = \int_0^a \int_0^{2\pi} 2 \cdot (x \cos \theta + a - a \sin \theta) d\theta dx$$

$$= 2a \cdot [2\pi \cdot x^2/2]_0^a = 2\pi a^3$$

$$\begin{aligned}
 \text{RHS} &= \oint_C F \cdot d\mathbf{r} = \oint_C [(y^2 - x^2)i + (x^2 - y^2)j] \cdot (dx i + dy j) \\
 &= \oint_C (x^2 - y^2) (-dx + dy) \\
 &\quad \text{put } x = a + a \cos \theta, y = a \sin \theta \\
 &\quad [\because C \text{ is } (x-a)^2 + y^2 = a^2] \\
 &= \int_0^{2\pi} (a^2(1+\cos^2\theta) - a^2\sin^2\theta) (+a\sin\theta + a\cos\theta) d\theta \\
 &= a^3 \int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta - \sin^2\theta) (\sin\theta + \cos\theta) d\theta \\
 &= a^3 \int_0^{2\pi} 2\cos\theta (1 + \cos\theta) (\sin\theta + \cos\theta) d\theta \\
 &= a^3 \int_0^{2\pi} 2\cos\theta (\sin\theta + \sin\theta\cos\theta + \cos\theta + \cos^2\theta) d\theta \\
 &= 2a^3 \times [0 + 0 + \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) d\theta + \int_0^{2\pi} \frac{1}{4}(4\cos^2\theta + 2\cos 2\theta) d\theta] \\
 &= 2a^3 [\pi] = 2a^3 \pi = \text{LHS}
 \end{aligned}$$

*** Hence stated theorem verified.

END OF THE EXAMINATION

ROUGH SPACE

ROUGH SPACE

$$\begin{aligned}
 f(uv) &= ufv - \int uv du \\
 &= \left[e^{-x} - \frac{1}{x^2} \right] - \int e^{-x} - \frac{1}{x^2} dx \\
 a^2b^2 &= (a-b)(a^2+b^2+ab) \\
 a^2(b^2) &= (a-b)(a^2+b^2+ab) \\
 a^2(a^2) &= a^2(b^2) \\
 -a(b^2) &= -a(b^2) \\
 +2 = 0 &= +2 = 0 \\
 a^2 - 2a &= 0 \\
 2a^2 - 5a + 2 &= 0 \\
 a = 2 &= 2
 \end{aligned}$$