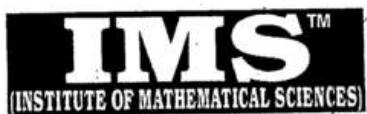


A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



130
250

TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-I(M) IAS / T-09

MATHEMATICS

by K. VENKANNA

The person with 14 years of Teaching Experience

FULL TEST P-I

Time: Three Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 52 pages and has 34 PART / SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated.
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name

Vaishnavi Guntupalli

Roll No:

Test Centre

Hyderabad

Medium

English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Gaurav

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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THIS SPACE**

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INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			05
	(b)			02
	(c)			08
	(d)			08
	(e)			
2	(a)			16
	(b)			4
	(c)			23
	(d)			17
3	(a)			36
	(b)			24
	(c)			1
	(d)			60
4	(a)			38
	(b)			98
	(c)			40
	(d)			
5	(a)			138
	(b)			
	(c)			
	(d)			
	(e)			
6	(a)			06
	(b)			08
	(c)			11
	(d)			13
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			10
	(b)			08
	(c)			06
	(d)			16
Total Marks				138/250

138
250

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SECTION-A

1. (a) If we shift to $A - 7I$, what are the eigenvalues and eigenvectors and how are they related to those of A ?

$$B = A - 7I = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix} \quad (10)$$

$$\text{Given } B = A - 7I = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$$|B - \lambda I| = 0 \Rightarrow \begin{vmatrix} -6-\lambda & -1 \\ 2 & -3-\lambda \end{vmatrix} = 0 \quad |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+6)(\lambda+3) + 2 = 0 \quad \& \quad (\lambda-1)(\lambda-4) + 2 = 0$$

$$\Rightarrow \lambda^2 + 9\lambda + 20 = 0 \quad \& \quad \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda+4)(\lambda+5) = 0 \quad \& \quad (\lambda-3)(\lambda-2) = 0$$

$$\Rightarrow \lambda = -4, -5 \quad \& \quad \lambda = 2, 3$$

$$\Rightarrow \text{Eigen values of } A = \underline{\underline{2, 3}}$$

$$\text{Eigen values of } B = \underline{\underline{-4, -5}}$$

Consider $BX = \lambda X$ where λ is eigen value of B
 λX is eigen vector corresponding to λ

$$\Rightarrow (A - 7I)X = \lambda X$$

$$\Rightarrow AX = (\lambda + 7)X$$

$$\Rightarrow \text{Eigen values of } A = 7 + \text{Eigen values of } B$$

& Eigen vectors of A & B are same

$$\text{For } A, \lambda = 2 \Rightarrow \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x-y \\ 2x+2y \end{pmatrix}$$

$$\Rightarrow x = 0, y = 0$$

$$\text{For } A, \lambda = 3 \Rightarrow \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x-y \\ 2x+y \end{pmatrix}$$

$$\Rightarrow 5x = -y ; \quad \cancel{x = y} \Rightarrow x = 0, y = 0$$

$$\Rightarrow \text{Eigen vector of } A \& B = \underline{\underline{(0, 0)}}$$

- (b) Let V be the set of all complex-valued functions f on the real line such that (for all t in \mathbb{R})

$$f(-t) = \overline{f(t)}.$$

the bar denotes complex conjugation. Show that V , with operations $(f+g)(t) = f(t)+g(t)$
 $(cf)(t) = cf(t)$.

is a vector space over the field of real numbers. (10)

Clearly V is a vector space over \mathbb{R} if

$$(f+g)(t) = f(t) + g(t)$$

$$\& (cf)(t) = cf(t)$$

~~Consider $(f+g)(t)$~~

$$\text{Consider } (f+g)(t) = \overline{(f+g)(-t)} \quad [\because f(-t) = \overline{f(t)}]$$

$$= \overline{f(-t) + g(-t)}$$

$$= \overline{f(-t)} + \overline{g(-t)}$$

$$= f(t) + g(t)$$

Now consider $(cf)(t) = \overline{(cf)(-t)} \quad [\because f(-t) = \overline{f(t)}]$

-02-

~~$(cf)(t)$~~

$$= \overline{c f(-t)}$$

$$= c \overline{f(-t)} \quad [\because c \text{ is real}]$$

$$= c f(t)$$

$\Rightarrow V$ is a vector space over \mathbb{R} .

(c) Determine the values of A and B for which

$$\lim_{x \rightarrow 0} \frac{\sin 3x + A \sin 2x + B \sin x}{x^5}$$
 exists and find the limit

(10)

We know that

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \dots$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin 3x + A \sin 2x + B \sin x}{x^5}$$

$$= \lim_{x \rightarrow 0} \frac{\left\{ 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} \right\} + A \left\{ 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} \right\} + B \left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!} \right\}}{x^5}$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{1}{5!} (3^5 + A \cdot 2^5 + B) - \frac{1}{3! x^2} (3^3 + A \cdot 2^3 + B) + \frac{1}{x^4} (3 + 2A + B) \right\}$$

For the limit to exist,

$$3+2A+B=0$$

$$\& 3^3+A \cdot 2^3+B=0$$

$$\Rightarrow 24+6A=0 \Rightarrow A=-4.$$

$$\& B=5$$

$$\Rightarrow \underline{A = -4, B = 5}$$

$$\& \text{Limit} = \frac{1}{5!} (3^5 + A \cdot 2^5 + B)$$

$$= \frac{1}{120} (120)$$

$$= 1$$



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(d) Investigate the maxima and minima of

$$f(x,y) = x^2 + 3xy + y^2 + x^3 + y^3 \quad \text{--- (1)}$$

(10)

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x + 3y + 3x^2 = 0 \quad \text{--- (2)}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 3x + 2y + 3y^2 = 0 \quad \text{--- (3)}$$

From (2) & (3),

$$x - y + 3(y^2 - x^2) = 0$$

$$\Rightarrow (x-y)(x+y) \quad x=y \quad \text{or} \quad x+y = \frac{1}{3}$$

$$x=y \Rightarrow 5x + 3x^2 = 0 \Rightarrow x=0, -\frac{5}{3}$$

$$\text{i.e. } A(0,0), B\left(-\frac{5}{3}, -\frac{5}{3}\right)$$

$$x+y = \frac{1}{3} \text{ & } 3(x+y) + 3x^2 - x = 0$$

$$\Rightarrow 1 + 3x^2 - x = 0$$

$$\Rightarrow (3x-1)(x+1)=0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1+12}}{6} \Rightarrow x \text{ is not real.}$$

\Rightarrow Stationary points of (1) are $A(0,0)$ & $B\left(-\frac{5}{3}, -\frac{5}{3}\right)$

$$\frac{\partial^2 f}{\partial x^2} = 2 + 4x \quad (A) \quad ; \quad \frac{\partial^2 f}{\partial y^2} = 2 + 4y; \quad \frac{\partial^2 f}{\partial x \partial y} = 3 \quad (B)$$

At $A(0,0)$, $A = 2 > 0$

$$\Delta AC - B^2 = 4 - 9 < 0$$

\Rightarrow At $A(0,0)$, f has an inflection point

$$\text{At } B\left(-\frac{5}{3}, -\frac{5}{3}\right), A = 2 - \frac{20}{3} < 0; \quad AC - B^2 = \left(\frac{14}{3}\right)^2 - 9 > 0$$

\Rightarrow At $B\left(-\frac{5}{3}, -\frac{5}{3}\right)$, f has a maxima.

(e) Find the angle between the lines given by $x+y+z=0$ and

$$\frac{yz}{q-r} + \frac{zx}{r-p} + \frac{xy}{p-q} = 0.$$

(10)

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3. (a) (i) If a 3 by 3 upper triangular matrix has diagonal entries 1, 2, 7 how do you know it can be diagonalized? What is Λ ? When Λ an eigen value matrix. (22)
- (ii) Let V be the vector space of all real polynomials. Consider the subspace W spanned by t^2+t+2 , t^2+2t+5 , $5t^2+3t+4$ and $2t^2+2t+4$. Then, find the dimension of W .

$$(ii) W = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & 2 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 5R_1 \\ R_4 \rightarrow R_4 + 2R_1 \end{array}$$

$$0 \neq \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_3 \rightarrow R_3 + 2R_2 \end{array}$$

⇒ Dimension of $W = 2$

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- 3 (b) Find the maximum value of $f(x, y, z) = x^2 + 2y - z^2$
subject to the constraints $2x-y=0$ and $y+z=0$

(18)

$$\text{Let } \phi(x, y, z) = x^2 + 2y - z^2 + \lambda_1(2x-y) + \lambda_2(y+z) \quad \text{--- (1)}$$

$$\frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + 2\lambda_1 = 0 \quad \text{--- (2)}$$

$$\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2 - \lambda_1 + \lambda_2 = 0 \quad \text{--- (3)}$$

$$\frac{\partial \phi}{\partial z} = 0 \Rightarrow -2z + \lambda_2 = 0 \quad \text{--- (4)}$$

$$\frac{\partial \phi}{\partial \lambda_1} = 0 \Rightarrow 2x = y \quad \text{--- (5)}$$

$$\frac{\partial \phi}{\partial \lambda_2} = 0 \Rightarrow y+z=0 \quad \text{--- (6)}$$

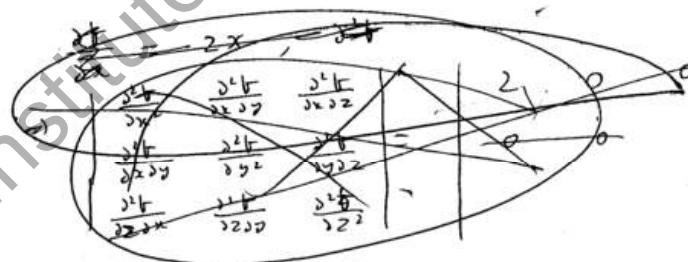
$$\Rightarrow \lambda_1 = -x, \lambda_2 = 2z \quad [\text{From (2) \& (4)}]$$

$$\Rightarrow 2 + x - 2z = 0 \quad \text{--- (7)} \quad [\text{From (3)}]$$

From (5), (6), (7)

$$2x = y, \quad x + 2y + 2z = 0 \Rightarrow 5x = -2$$

$$\Rightarrow x = -\frac{2}{5}, \quad y = -\frac{4}{5}, \quad z = \frac{4}{5}$$



\Rightarrow Maximum value of $f(x, y, z)$

is at $(-\frac{2}{5}, -\frac{4}{5}, \frac{4}{5})$

$$\text{Max value} = \left(-\frac{2}{5}\right)^2 + 2 \times \left(-\frac{4}{5}\right)^2 - \left(\frac{4}{5}\right)^2$$

$$\begin{aligned} &= \frac{4}{25} + 2 \times \frac{16}{25} - \frac{16}{25} \\ &= \frac{20}{25} \\ &= \frac{4}{5} \end{aligned}$$

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- 3 (c) Find the surface generated by (or find the locus of) straight lines drawn through a fixed point (α, β, γ) at right angles to their polar with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$. (15)

Polar of $P(\alpha, \beta, \gamma)$ w.r.t. $ax^2 + by^2 + cz^2 = 1$ is

$$\alpha x + b\beta y + c\gamma z = 1 \quad \text{--- (1)}$$

Any line through $(\alpha, \beta, \gamma) \perp^{\lambda}$ to (1) is given by

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$$

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SECTION-B

5. (a) Solve $(3 + 2 \sin x + \cos x) dy = (1 + 2 \sin y + \cos y) dx$.

(10)

$$\Rightarrow \frac{dx}{3 + 2 \sin x + \cos x} = \frac{dy}{1 + 2 \sin y + \cos y}$$

$$2\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2 + 2 \cos^2 \frac{x}{2} = \frac{dy}{1 + 2 \sin y + \cos y}$$

- (b) Solve $d^2y/dx^2 + 2(dy/dx) + 10y + 37 \sin 3x = 0$ and find the value of y when $x = \pi/2$ if it given that $y=3$ and $dy/dx=0$ when $x=0$. (10)

$$\text{Given } (D^2 + 2D + 10)y = -37 \sin 3x \quad \text{--- (1)}$$

Auxiliary eqn is $m^2 + 2m + 10 = 0$

$$\Rightarrow (m+1)^2 = -9$$

$$\Rightarrow m = -1 \pm 3i$$

$$\Rightarrow CF = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) \quad \text{--- (2)}$$

$$PI = \frac{1}{D^2 + 2D + 10} (-37 \sin 3x)$$

$$= -37 \cdot \frac{1}{-9 + 2D + 10} \sin 3x$$

$$= -37 \cdot \frac{1}{(2D+1)} \sin 3x$$

$$= -37 \cdot \frac{2D-1}{4-(-9)-1} \sin 3x$$

$$= 6 \cos 3x - \sin 3x \quad \text{--- (3)}$$

\Rightarrow Soln of (1) is

$$y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + 6 \cos 3x - \sin 3x$$

given $y=3$ & $Dy=0$, at $x=0$

$$\Rightarrow 3 = c_1 + 6 \Rightarrow c_1 = -3 \quad \text{--- (4)}$$

$$Dy = e^{-x}(-3c_1 \sin 3x + 3c_2 \cos 3x) - e^{-x}(c_1 \cos 3x + c_2 \sin 3x) \\ = -18 \sin 3x - 3 \cos 3x$$

$$\Rightarrow 0 = (3c_2) - (c_1) - 3 \Rightarrow 3c_2 = 0 \Rightarrow c_2 = 0$$

$$\Rightarrow y = -3e^{-x} \cos 3x + 6 \cos 3x - \sin 3x$$

$$y(\pi/2) = -\sin \frac{3\pi}{2} = 1$$

- (c) A uniform beam of length $2a$, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1} (b/a)^{1/3}$ (10)

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(d) A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3} \sqrt{\left(\frac{2a}{g}\right)} \left[\left(1 + \frac{h}{a}\right)^{1/2} - 1 \right].$$

Where a is the radius of the earth.

(10)

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(e) Prove the Frenet-Serret formula. (i) $\frac{dT}{dS} = \kappa N$ (ii) $\frac{dB}{dS} = \tau N$ (iii) $\frac{dN}{dS} = \tau B - \kappa T$ (10)

We know that

$$\bar{T} \cdot \bar{T} = 1$$

Differentiating this w.r.t s ,

$$\frac{d\bar{T}}{ds} \cdot \bar{T} = 0$$

$$\Rightarrow \frac{d\bar{T}}{ds} \text{ is } \perp^{\lambda} \text{ to } \bar{T} - \textcircled{1}$$

We know that $\text{curv} \& \text{ osculating plane is}$

$$[\bar{x}', \bar{x}'', \bar{x}'''] = 0 \quad \cancel{\text{---}}$$

$$\therefore \frac{d\bar{T}}{ds} = \frac{d}{ds}(\bar{x}') = \bar{x}''$$

$$\Rightarrow \frac{d\bar{T}}{ds} \text{ lies in osculating plane} - \textcircled{2}$$

$$\Rightarrow \text{From } \textcircled{1} \& \textcircled{2}, \frac{d\bar{T}}{ds} \text{ is along } \bar{n}$$

$$\text{Let } \frac{d\bar{T}}{ds} = \lambda \bar{n} - \textcircled{3}$$

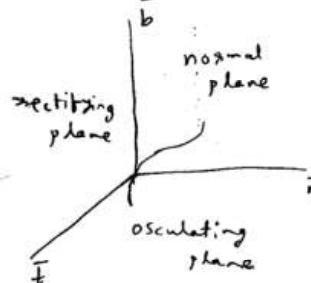
$$\text{We know that } \text{curvature, } k = \frac{d\bar{T}}{ds} - \textcircled{4}$$

$$\Rightarrow \text{From } \textcircled{3} \& \textcircled{4}, \therefore \left| \frac{d\bar{T}}{ds} \right| = \lambda |\bar{n}| = \lambda = k$$

$$\text{we have } \frac{d\bar{T}}{ds} = k \bar{n} - \textcircled{I}$$

$$\text{Now consider } \bar{b} \cdot \bar{b} = 1 \Rightarrow \frac{d\bar{b}}{ds} \cdot \bar{b} = 0$$

$$\Rightarrow \frac{d\bar{b}}{ds} \text{ is } \perp^{\lambda} \text{ to } \bar{b} - \textcircled{S}$$



$$\text{Also } \bar{b} \cdot \bar{t} = 0 \Rightarrow \bar{b} \cdot \frac{d\bar{t}}{ds} + \bar{t} \cdot \frac{d\bar{b}}{ds} = 0$$

$$\Rightarrow \bar{b} \cdot k\bar{n} + \bar{t} \cdot \frac{d\bar{b}}{ds} = 0 \quad [\text{from 1}]$$

$$\Rightarrow \bar{t} \cdot \frac{d\bar{b}}{ds} = 0$$

$$\Rightarrow \frac{d\bar{b}}{ds} \text{ is } \perp^{\lambda} \text{ to } \bar{t} - \textcircled{6}$$

From ⑤ & ⑥, $\frac{d\bar{b}}{ds}$ is along \bar{n}

Let $\frac{d\bar{b}}{dt} = v\bar{n}$; but since we know that
torsion $\tau = \frac{d\bar{b}}{ds}$, we have $v = \tau$

$$\Rightarrow \frac{d\bar{b}}{dt} = -\tau\bar{n} - \textcircled{II}$$

$$\bar{n} = \bar{b} \times \bar{t} \Rightarrow \frac{d\bar{n}}{ds} = \bar{b} \times \frac{d\bar{t}}{ds} + \frac{d\bar{b}}{ds} \times \bar{t}$$

$$\Rightarrow \frac{d\bar{n}}{ds} = \bar{b} \times (k\bar{n}) + (-\tau\bar{n}) \times \bar{t} = \tau\bar{b} - k\bar{t} - \textcircled{III}$$

6. (a) Find the solution of the differential equation $y = 2xp - y^2$ where $p = dy/dx$. Also find the singular solution. (10)

$$\text{Given } y = 2xp - y^2 - \textcircled{1}$$

$$\Rightarrow y = \frac{2xp}{1+y^2} - \textcircled{2}$$

b) Differentiate ② w.r.t. x

$$\Rightarrow p = \frac{2p}{1+p^2} + 2x \cdot \left(\frac{1+p^2 - p \cdot (2p)}{(1+p^2)^2} \right) \frac{dp}{dx}$$

$$\Rightarrow p = \frac{2p}{1+p^2} + \frac{2x(1-p^2)}{(1+p^2)^2} \frac{dp}{dx}$$

case try again

$$\Rightarrow p - \frac{2p}{1+p^2} = \frac{2x(1-p^2)}{(1+p^2)^2} \frac{dp}{dx}$$

$$\Rightarrow p + p^3 - 2p = \frac{2x(1-p^2)}{1+p^2} \frac{dp}{dx}$$

$$\Rightarrow p(p^2 - 1) = \frac{2x(1-p^2)}{1+p^2} \frac{dp}{dx}$$

$$\begin{aligned} \Rightarrow \frac{dx}{2x} &= \frac{-p}{1+p^2} dp \\ \Rightarrow \ln x &= -\ln(1+p^2) + C_1 \\ \Rightarrow x(1+p^2) &= C \quad \text{where } C \text{ is an arbitrary constant} \\ \Rightarrow p &= \sqrt{\frac{C}{x}-1} \Rightarrow y = \int \sqrt{\frac{C}{x}-1} dx \end{aligned}$$

Singular soln: $2p^2 - 2xp + y = 0$

$$\begin{aligned} \Rightarrow p - \text{disc} &= 4x^2 - 4y^2 = 4(x^2 - y^2) \\ \text{Put } x^2 = y^2 \text{ in } ① &\Rightarrow p = \frac{x}{y} \\ \Rightarrow LHS = y &= 2x - 2y - 2 \cdot (\frac{x}{y})^2 = \frac{x^2}{y} = \frac{y^2}{y} = y \end{aligned}$$

$\Rightarrow x^2 - y^2 = 0$ satisfies ①

\Rightarrow singular soln. of ① is $x^2 - y^2 = 0$

- (b) Prove that the orthogonal trajectories of the family of conics $y^2 - x^2 + 4xy - 2cx = 0$ consist of a family of cubics with the common asymptote $x + y = 0$. (12)

Given family of conics is

$$y^2 - x^2 + 4xy - 2cx = 0 \quad -①$$

Differentiating ① w.r.t. x ,

$$2y \frac{dy}{dx} - 2x + 4\left(x \frac{dy}{dx} + y\right) - 2c = 0 \quad -②$$

Put c from ② in ① to get get ode for ①

$$\text{i.e. } y^2 - x^2 + 4xy - x(2y \frac{dy}{dx} - 2x + 4(x \frac{dy}{dx} + y)) = 0$$

$$\Rightarrow x(4x+2y)p = y^2 + 8x^2$$

$$\Rightarrow p = \frac{x^2 + y^2}{x(4x+2y)} \quad -③$$

Orthogonal trajectories are given by
replacing p with $-1/p$ in ③

$$\Rightarrow -\frac{1}{p} = \frac{x^2+y^2}{2x(2x+y)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x(2x+y)}{x^2+y^2}$$

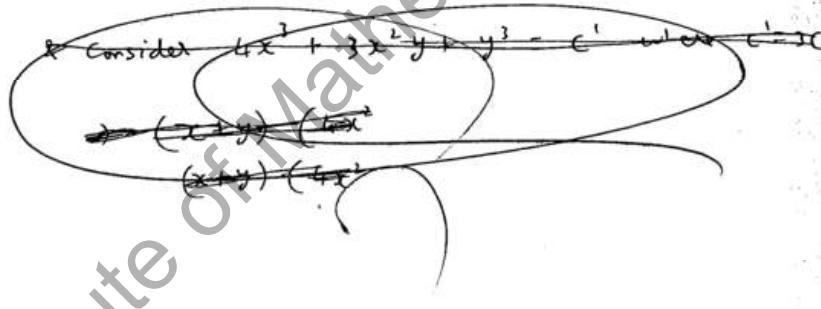
$$\Rightarrow 2x(2x+y)dx + (x^2+y^2)dy = 0$$

$$\Rightarrow 4x^2dx + (2xydx + x^2dy) + y^2dy = 0$$

$$\Rightarrow \frac{4}{3}x^3 + x^2y + \frac{1}{3}y^3 = C \quad \text{--- (4)}$$

where C is an arbitrary constant.

Clearly the orthogonal trajectories family
is a family of cubics.



- (c) Apply the method of variation of parameters to solve $y''+4y=\sin^2 x$ (13)

$$\text{Given } y''+4y=\sin^2 x \quad \textcircled{1}$$

Taking $y''+4y=0$, we get the solutions

as $\sin 2x, \cos 2x$

$$\text{Let } y_1(x) = \sin 2x, \quad y_2(x) = \cos 2x \quad \textcircled{2}$$

Let solution of $\textcircled{1}$ be $y = v_1(x) \cdot y_1(x) + v_2(x) \cdot y_2(x)$

$$w = \begin{vmatrix} \sin 2x & \cos 2x \\ -2\cos 2x & 2\sin 2x \end{vmatrix} = 2 \quad \textcircled{3}$$

$$v_1(x) = \int -\frac{y_2 R}{w} dx \quad \& \quad v_2(x) = \int \frac{y_1 R}{w} dx$$

$$\Rightarrow V_1(x) = \int -\frac{1}{2} \cos 2x \sin^2 x \, dx ; \quad V_2(x) = \int \frac{1}{2} \sin 2x \sin^3 x \, dx$$

$$\Rightarrow V_1(x) = -\frac{1}{2} \int \cos 2x \cdot \left(\frac{1-\cos 2x}{2} \right) dx$$

$$= -\frac{1}{4} \int \left\{ \cos 2x - \frac{1}{2} (1 + \cos 2x) \right\} dx$$

$$= -\frac{1}{4} \left\{ \frac{1}{2} \sin 2x - \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right) \right\} + C_1$$

$$= \frac{1}{8} \left(\frac{1}{4} \sin 4x + x - \sin 2x \right) + C_1$$

$$V_2(x) = \frac{1}{2} \int 2 \cos x \sin^3 x \, dx$$

$$= \frac{1}{4} \sin^4 x + C_2$$

\Rightarrow Solution of ① is

$$y = C_1 \sin 2x + C_2 \cos 2x + \frac{1}{8} \sin 2x \left(\frac{1}{4} \sin 4x + x - \sin 2x \right) \\ + \frac{1}{4} \cos 2x \sin^4 x$$

- (d) By using Laplace transform method, solve the differential equation $(D^2 + 9)y = 18t$, with $y(0)=0$, $y(\pi/2)=0$. (15)

$$\text{Given } (D^2 + 9)y = 18t \quad \text{--- (1)} \quad y(0) = 0, \quad y(\pi/2) = 0$$

Applying Laplace transform to (1),

$$s^2y(s) - sy(0) - y'(0) + 9y(s) = 18 \cdot \frac{1}{s^2}$$

$$\text{Let } y'(0) = C$$

$$\Rightarrow (s^2 + 9)y(s) = \cancel{\frac{18}{s^2}} + C$$

$$\Rightarrow y(s) = \frac{18}{s^2(s^2 + 9)} + \frac{C}{s^2 + 9}$$

$$= \frac{2}{s^2} + \frac{-2}{s^2 + 9} + \cancel{\frac{C}{s^2 + 9}}$$

$$\Rightarrow y(t) = 2t + (C-2) \sin 3t$$

$$\text{Given } y(\pi/2) = 0$$

$$\Rightarrow 0 = \pi + (C-2) \times 8(-1)$$

$$\Rightarrow C = \pi + 2$$

~~$$\Rightarrow y(t) = 2t + \pi \sin 3t$$~~

- (c) A shot fired with velocity V at an elevation θ strikes a point P on the horizontal plane through the point of projection. If the point P is receding from the gun with velocity v , show that the elevation must be changed to ϕ , where

$$\sin 2\phi = \sin 2\theta + \frac{2v}{V} \sin \phi. \quad (16)$$

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P.T.O.

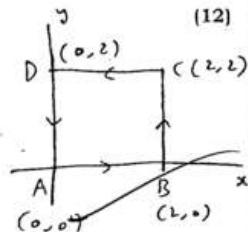
8. (a) Verify Green's theorem in the plane for $\int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy$, where C is the square with vertices (0,0), (2,0), (2,2), (0,2).

Green's theorem is given by

$$\int_C P dx + Q dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$LHS = \int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy$$

$$\begin{aligned}
 &= \int_{x=0}^2 x^2 dx + \int_{y=0}^2 (y^2 - 4y) dy + \int_{x=2}^0 (x^2 - 8x) dx + \int_{y=2}^0 y^2 dy \\
 &\quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\
 &= \left. \frac{x^3}{3} \right|_0^2 + \left. \left(\frac{y^3}{3} - 2y^2 \right) \right|_0^2 + \left. \left(\frac{x^3}{3} - 4x^2 \right) \right|_2^0 + \left. \frac{y^3}{3} \right|_2^0
 \end{aligned}$$



P.T.O.

$$= \frac{8}{3} + \left(\frac{8}{3} - 8 \right) + \left(-\frac{8}{3} + 16 \right) - \frac{8}{3}$$

$$= 16 - 8 = \underline{\underline{8}}$$

$$\begin{aligned} RHS &= \iint_S \left\{ \frac{\partial}{\partial x} (y^2 - 4xy) - \frac{\partial}{\partial y} (x^2 - xy^3) \right\} dx dy \\ &= \int_0^2 \int_0^2 (-2y + 3xy^2) dx dy \\ &= \int_0^2 \left(-y^2 + xy^3 \right) \Big|_0^2 dx \\ &= \int_0^2 (-4 + 8x) dx \\ &= -4x + 4x^2 \Big|_0^2 \\ &= -8 + 16 \\ &= \underline{\underline{8}} = LHS \end{aligned}$$

Hence Green's theorem is verified.

- (b) (i) Show that $\vec{q} \cdot \nabla \vec{q} = \frac{1}{2} \nabla q^2 - \vec{q} \times \text{curl } \vec{q}$
 $\text{div. curl } (\text{curl } \vec{a} \phi) + \nabla^2 \text{div}(\vec{a} \phi) = \vec{a} \cdot \text{grad } \nabla^2 \phi$ where ϕ is a scalar point function. (12)

(i) We know that

$$\cancel{\text{grad}} \text{ grad } (A \cdot B) = (A \cdot \nabla) B + (B \cdot \nabla) A + A \times \text{curl } B + B \times \text{curl } A$$

put $A = B = \vec{q}$

$$\Rightarrow \nabla(\vec{q}^2) = (\vec{q} \cdot \nabla)\vec{q} + (\vec{q} \cdot \nabla)\vec{q} + \vec{q} \times \text{curl } \vec{q} + \vec{q} \times \text{curl } \vec{q}$$

$$\Rightarrow \underline{(\vec{q} \cdot \nabla)\vec{q} = \frac{1}{2} \nabla(\vec{q}^2) - \vec{q} \times \text{curl } \vec{q}}$$

(ii) LHS = $\nabla \cdot (\nabla \times \text{curl } \vec{a} \phi) + \nabla^2 \text{div}(\vec{a} \phi)$

We know that

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\cancel{\nabla \cdot (\vec{A} \phi)} = \nabla \phi \cdot \vec{A} + \phi \nabla \cdot \vec{A}$$

$$\cancel{\text{LHS}} = \nabla \cdot (\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A})$$

$$\Rightarrow \text{LHS} = \nabla \cdot (\nabla(\nabla \cdot \vec{a} \phi) - \nabla^2(\vec{a} \phi)) + \nabla^2 \text{div}(\vec{a} \phi)$$

$$= 2 \nabla^2 \text{div}(\vec{a} \phi) - \cancel{\nabla \cdot \nabla^2(\vec{a} \phi)}$$

$$= 2 \nabla^2 \{ \nabla \phi \cdot \vec{a} + \phi \nabla \cdot \vec{a} \} - \nabla \cdot \nabla^2(\vec{a} \phi)$$

=

- (c) If $\phi = 2xyz^2$, $F = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the Curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals. (i) $\int_C \phi d\lambda$ (ii) $\int_C F \cdot d\lambda$ (08)

$$\begin{aligned}
 (i) \int_C \phi d\lambda &= \int_C \phi \frac{dx}{dt} dt \\
 &= \int_0^1 2 \cdot t^2 \cdot 2t \cdot (t^3)^2 \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\
 &= \int_0^1 4t^9 \sqrt{(2t)^2 + 2^2 + (3t^2)^2} dt \\
 &= \int_0^1 4t^9 \sqrt{4 + 4t^2 + 9t^4} dt
 \end{aligned}$$

$$\begin{aligned}
 (ii) \int_C \phi d\lambda &= \int_C 2xyz^2 \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\
 &= \int_0^1 2 \cdot (t^2) \cdot (2t) \cdot (t^3)^2 \cdot (2t\hat{i} + 2\hat{j} + 3t^2\hat{k}) dt \\
 &= \int_0^1 4t^9 (2t\hat{i} + 2\hat{j} + 3t^2\hat{k}) dt \\
 &= 4 \left[2 \cdot \frac{1}{11} i + 2 \cdot \frac{1}{10} j + 3 \cdot \frac{1}{12} k \right] = 4 \left[\frac{2}{11} i + \frac{1}{5} j + \frac{1}{4} k \right] \\
 &\quad \xrightarrow{\text{Crossed out}} \left[\frac{120 + 132 + 3 \times 55}{11 \times 60} \right] = \frac{417}{165}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \int_C F \cdot d\lambda &= \int_0^1 (2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}) \cdot (2t\hat{i} + 2\hat{j} + 3t^2\hat{k}) dt \\
 &= \int_0^1 \{ (-3t^5 - 2t^4)\hat{i} + (2t^5 - 6t^5)\hat{j} + (4t^3 + 2t^4)\hat{k} \} dt \\
 &= \left(-\frac{3}{6} - \frac{2}{5} \right) \hat{i} - \frac{4}{6} \hat{j} + \left(\frac{4}{4} + \frac{2}{5} \right) \hat{k} \\
 &= -\frac{9}{10} \hat{i} - \frac{2}{3} \hat{j} + \frac{7}{5} \hat{k}
 \end{aligned}$$

- (d) Verify the divergence theorem for $\mathbf{A} = 2x^2 y \hat{i} - y^2 \hat{j} + 4xz^2 \hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and $x=2$. (18)

Divergence theorem is given by

$$\iint_S \mathbf{A} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \mathbf{A} \, dv$$

where V is volume enclosed by surface S

$$RHS = \iiint_V \nabla \cdot \mathbf{A} \, dv$$

$$= \iiint_V \nabla \cdot (2x^2 y \hat{i} - y^2 \hat{j} + 4xz^2 \hat{k}) \, dv$$

$$= \iiint_V (4xy - 2y + 8xz) \, dv$$

$$= \int_{\rho=0}^3 \int_{\theta=0}^{\pi/2} \int_{x=0}^2 (4x \cdot \rho \cos \theta - 2\rho \sin \theta + 8x \cdot \rho \sin \theta) \, \rho \, dx \, d\rho \, d\theta$$

$$= \int_{\rho=0}^3 \int_{\theta=0}^{\pi/2} (2x^2 \rho \cos \theta - 2\rho^2 \sin \theta + 4x^2 \rho \sin \theta) \Big|_0^2 \, \rho \, d\rho \, d\theta$$

$$= \int_{\rho=0}^3 \int_{\theta=0}^{\pi/2} \rho^2 \cdot \rho \, d\rho \cdot (8 \cos \theta - 4 \cos \theta + 16 \sin \theta) \, d\theta$$

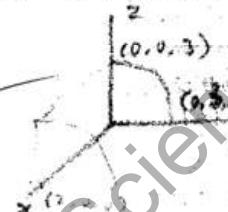
$$= \frac{\rho^3}{3} \Big|_0^3 \cancel{(8 \sin \theta - 16 \cos \theta)} \Big|_0^{\pi/2}$$

$$= \frac{27}{3} \times (4 - (-16)) = \underline{\underline{180}}$$

$$L.H.S. = \iint_S \mathbf{A} \cdot \hat{n} \, ds$$

$$= \iint_{S_1} \mathbf{A} \cdot (-\hat{i}) \, ds + \iint_{S_2} \mathbf{A} \cdot (\hat{i}) \, ds + \iint_{S_3} \mathbf{A} \cdot \hat{n} \, ds$$

at $x=0$ at $x=2$ curved surface



For $y^2 + z^2 = 9$, $\nabla f = 2y\mathbf{j} + 2z\mathbf{k}$

$$\Rightarrow \hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4y^2 + 4z^2}} = \frac{1}{3}(y\mathbf{i} + z\mathbf{k})$$

$$\Rightarrow LHS = \iint_{S_1} -2x^2y \, ds + \iint_{S_2} 2x^2y \, ds + \iint_{S_3} \frac{1}{3}(-y^3 + 4xz^3) \, ds$$

$$= \iint_{S_2} 8y \, ds + \iint_{S_3} \frac{1}{3}(-y^3 + 4xz^3) \, ds$$

$$= 8 \int_{x=0}^3 \int_{\theta=0}^{\pi/2} 8x \cos \theta \cdot x \, dx \, d\theta + \frac{1}{3} \int_{\theta=0}^{\pi/2} \int_{x=0}^2 [(-3 \cos^3 \theta) + 4x(3 \sin \theta)^3] \, dx \, d\theta$$

$$= 8 \cdot \frac{x^3}{3} \Big|_0^3 \sin \theta \Big|_0^{\pi/2} + \frac{1}{3} \int_{\theta=0}^{\pi/2} \int_{x=0}^2 [27 \cos^3 \theta + 108 \sin^3 \theta \cdot x] \, 3d\theta \, dx$$

$$= 8 \times 9 + \frac{1}{3} \int_{\theta=0}^{\pi/2} (27 \times 2 \cos^3 \theta + 108 \times \frac{2^2}{2} \sin^3 \theta) \, d\theta$$

$$= 72 + 54 \times \left[-\frac{2}{3} \times 1 \right] + 108 \times 2 \times \left[\frac{2}{3} - 1 \right]$$

~~$$= 72 - 36 + 36 \times 4$$~~

$$= \frac{180}{2} = LHS$$

Hence, divergence theorem is verified.