

180/250

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A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-II(M) IAS / T-10

MATHEMATICS

by K. VENKANNA
The person with 14 years of Teaching Experience

FULL TEST P-II

Time: Three Hours

Maximum Marks: 250

INSTRUCTIONS

- This question paper-cum-answer booklet has 52 pages and has 36 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
- Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
- A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated.
- Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
- Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
- The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
- Symbols/notations carry their usual meanings, unless otherwise indicated.
- All questions carry equal marks.
- All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
- All rough work should be done in the space provided and scored out finally.
- The candidate should respect the instructions given by the invigilator.
- The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name

Nitish-k

Roll No.

149709

Test Centre

Bangalore

Medium

English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Nitish-k

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			7
	(b)			7
	(c)			5
	(d)			7
	(e)			8
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			8
	(b)			14
	(c)			16
	(d)			
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			8
	(b)			7
	(c)			9
	(d)			7
	(e)			
6	(a)			5
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7	(a)			18
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8	(a)			
	(b)			
	(c)			
	(d)			
Total Marks				

IMS Institute of Mathematical Sciences

SECTION-A

1. (a) If $|a^5| = 12$, what are the possibilities for $|a|$? If $|a^4| = 12$, what are the possibilities for $|a|$? (10)

$$o(a^5) = 12 \Rightarrow (a^5)^{12} = e \Rightarrow \underline{a^{60} = e}$$

$$\therefore o(a) \mid 60$$

$$\Rightarrow \underline{o(a) = 1, 2, 3, 4, 5, 6, 10, 12, 15, 30, 60}$$

$$o(a^4) = 12 \Rightarrow (a^4)^{12} = e \Rightarrow \underline{a^{48} = e}$$

$$o(a) \mid 48$$

$$\underline{o(a) = 1, 2, 3, 4, 6, 8, 12, 16, 24, 48}$$

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(b) Give addition and multiplication tables for $\mathbb{Z}/8\mathbb{Z}$. Are $\mathbb{Z}/8\mathbb{Z}$ and \mathbb{Z}_4 isomorphic?

$$\frac{\mathbb{Z}}{8\mathbb{Z}} = \{0+8\mathbb{Z}, 2+8\mathbb{Z}, 4+8\mathbb{Z}, 6+8\mathbb{Z}\}$$

+	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$8\mathbb{Z}$	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$2+8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$	$8\mathbb{Z}$
$4+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$	$8\mathbb{Z}$	$2+8\mathbb{Z}$
$6+8\mathbb{Z}$	$6+8\mathbb{Z}$	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$

x	$0+8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$0+8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$
$2+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$
$4+8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$
$6+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$

Let $\phi: \mathbb{Z} \rightarrow \mathbb{Z}_4$

$$s.t. \phi(2n) = 2n \pmod{4}$$

clearly ϕ is onto homomorphism.

Let $2x \in \text{Ker } \phi \Leftrightarrow \phi(2x) = 0 \pmod{4}$

$$\Leftrightarrow 2x \pmod{4} = 0 \pmod{4}$$

$$\Leftrightarrow 4 \mid 2x$$

$$\Leftrightarrow 8 \mid 2x$$

$$\Leftrightarrow 2x = 8k ; k \in \mathbb{Z}$$

SECTION-A

1. (a) If $|a^5| = 12$, what are the possibilities for $|a|$? If $|a^4| = 12$, what are the possibilities for $|a|$? (10)

$$o(a^5) = 12 \Rightarrow (a^5)^{12} = e \Rightarrow \underline{a^{60} = e}$$

$$\therefore o(a) | 60$$

$$\Rightarrow \underline{o(a) = 1, 2, 3, 4, 5, 6, 10, 12, 15, 30, 60}$$

$$o(a^4) = 12 \Rightarrow (a^4)^{12} = e \Rightarrow \underline{a^{48} = e}$$

$$o(a) | 48$$

$$\underline{o(a) = 1, 2, 3, 4, 6, 8, 12, 16, 24, 48}$$

-07-

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$$\Rightarrow 2x \in 8Z$$

$$\therefore \text{ker } \phi = 8Z$$

$$\therefore \frac{2Z}{8Z} \cong Z_4$$

(c) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n^p (1+x^{2n})}$ is absolutely and uniformly convergent for all real x , if $p > 1$. (10)

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n x^{2n}}{n^p (1+x^{2n})} \right| = \sum_{n=1}^{\infty} \frac{x^{2n}}{n^p (1+x^{2n})}$$

$$= \sum_{n=1}^{\infty} f_n(x) g_n(x)$$

$$f_n(x) = \frac{1}{n^p} \quad \text{a) } \sum f_n(x) \text{ is convergent } \forall p > 1$$

$$g_n(x) = \frac{x^{2n}}{1+x^{2n}} = \frac{y^n}{1+y^n}; y = x^2 > 0$$

$$g_{n+1} - g_n(x) = y^n \left[\frac{y}{1+y^{n+1}} - \frac{1}{1+y^n} \right]$$

$$= y^n \frac{(y-1)}{(1+y^n)(1+y^{n+1})}$$

if $y > 1 \Rightarrow g_{n+1}(x) > g_n(x) \Rightarrow$ } In either case
 if $y < 1 \Rightarrow g_{n+1}(x) < g_n(x) \Rightarrow$ } $\langle g_n(x) \rangle$ is
 MONOTONIC.

\therefore By Abels Test.

$\sum b_n(x) g_n(x)$ is uniformly convergent

\therefore Given series is absolutely & uniformly convergent.

(d) Discuss the continuity of the following complex-valued function at $z=0$

$$f(z) = \begin{cases} \frac{1 - \exp(-|z|^2)}{|z|^2} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

(10)

$$f(x, y) = \frac{1 - e^{-(x^2+y^2)}}{x^2+y^2}; \quad x \neq 0, y \neq 0.$$

$e^{-x} \approx 1 - x$ for small x .

$$f(x, y) = \frac{1 - (1 - (x^2 + y^2))}{x^2 + y^2}$$

$$f(x, y) = \frac{x^2 + y^2}{x^2 + y^2} = 1$$

$$\therefore \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 1 = f(0, 0)$$

$\therefore f(x, y)$ is continuous at $(0, 0)$

$\Rightarrow f(z)$ is continuous at $z = 0$

(c) Consider the following problem:

$$\text{Maximise } Z = 2x_1 - 4x_2 + 5x_3 - 6x_4$$

$$\text{subject to } x_1 + 4x_2 - 2x_3 + 8x_4 \leq 2$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 \leq 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Determine: (i) The maximum number of possible basic solutions, (ii) The feasible extreme points (iii) The optimal basic feasible solution. (10)

$$m = 2, n = 4$$

$$\underline{6C_2 = 15}$$

maximum number of basic solution = $m = 2$

$n - m = 2$ non basic variable

first convert into standard form

$$x_1 = 0; x_2 = 0 \Rightarrow x_3 = 0; x_4 = 1/4 \Rightarrow z = -3/2$$

$$x_2 = 0; x_3 = 0 \Rightarrow x_3 = 0; x_4 = 1/4 \Rightarrow z = -3/2$$

$$x_2 = 0; x_4 = 0 \Rightarrow x_3 = 8/5; x_3 = -1/5 \Rightarrow z = 11/5$$

$$x_3 = 0; x_4 = 0 \Rightarrow x_1 = 0; x_2 = 1/2 \Rightarrow z = -2$$

$$x_1 = 0; x_2 = 0$$

$$x_1 = 0; x_4 = 0 \Rightarrow x_2 = 1/2; x_3 = 0 \Rightarrow z = -2$$

\therefore optimal solution

$$x_1 = 8/5; x_2 = 0; x_3 = -1/5; x_4 = 0$$

$$z_{\max} = 11/5$$

2. (a) (i) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{Z} \right\}$ and $S = \left\{ \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} \mid r, s, t \in \mathbb{Z}, s \text{ is even} \right\}$. If S is an ideal of R , what can you say about r and t ?

(ii) Show that $\mathbb{Z}_6[x]/(x^2+x+1)$ is not a field. (16)

3. (a) Let G be a group and let a be an element of G .
- (i) If $a^m = e$, what can we say about the order of a ?
- (ii) If G is an infinite group, what can you say about the number of elements of order 8 in the group? Generalize. (15)

$$\Rightarrow a^m = e \Rightarrow \underline{\underline{o(a) \mid m}}$$

of -

ii) if G is an infinite group, then

number of elements of order 8 = n .

$$\Rightarrow \phi(8) \mid n$$

$$\Rightarrow \underline{\underline{4 \mid n}}$$

$$\Rightarrow \underline{\underline{n = 4k; k \in \mathbb{Z}}}$$

\therefore number of elements of order 8
can be 4 or 8 or 16 or ...

(b) A function f is defined in $[0, 1]$ as follows:

$$f(x) = \begin{cases} \frac{1}{a^r} & \text{when } \frac{1}{a^r} < x \leq \frac{1}{a^{r-1}} \text{ for } r=1, 2, 3, \dots \\ 0 & \text{at } x=0 \end{cases} \quad f(x) = \frac{1}{a^{r-1}} ; \quad \frac{1}{a^r} < x \leq \frac{1}{a^{r-1}}$$

Where a is an integer greater than 2. Show that

$$\int_0^1 f(x) dx \text{ exists and is equal to } \frac{a}{a+1}. \quad (17)$$

$$\begin{aligned} f(x) &= \frac{1}{a^1} ; \\ &= \frac{1}{a^2-1} ; \\ &\vdots \\ &= \frac{1}{a^{r-1}} ; \end{aligned}$$

$$\begin{aligned} \frac{1}{a} < x \leq \frac{1}{a-1} \\ \frac{1}{a^2} < x \leq \frac{1}{a^2-1} \\ \frac{1}{a^r} < x \leq \frac{1}{a^{r-1}} \end{aligned}$$

clearly $f(x)$ is discontinuous only at $x = 0, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \dots$ which has only one limit point viz. 0.

$\therefore f(x)$ is Riemann integrable.

consider

$$\int_{\frac{1}{a^n}}^1 f(x) dx = \sum_{r=1}^n \int_{\frac{1}{a^r}}^{\frac{1}{a^{r-1}}} \left[\frac{1}{a^{r-1}} \right] dx$$

$$= \sum_{r=1}^n \frac{1}{a^{r-1}} \left[\frac{1}{a^{r-1}} - \frac{1}{a^r} \right]$$

$$\begin{aligned}
 &= \sum_{r=1}^n \left(\frac{1}{a^{r-1}}\right)^2 \left[\frac{a-1}{a}\right] \quad ; \text{ let } n \rightarrow \infty \\
 &= \left(\frac{a-1}{a}\right) \left[1 + \frac{1}{a^2} + \left(\frac{1}{a^2}\right)^2 + \left(\frac{1}{a^3}\right)^2 + \dots\right] \\
 &= \left(\frac{a-1}{a}\right) \left[1 + \frac{1}{a^2} + \left(\frac{1}{a^2}\right)^2 + \left(\frac{1}{a^2}\right)^3 + \dots\right] \\
 &= \left(\frac{a-1}{a}\right) \frac{1}{1 - \frac{1}{a^2}} \quad \because \text{as } a + ar + ar^2 + \dots = \frac{a}{1-r} \\
 &= \frac{a-1}{a} \cdot \frac{a^2}{a^2-1} = \frac{a}{a+1} //
 \end{aligned}$$

(c) Solve the following transportation problem.

(18)

	D ₁	D ₂	D ₃	a
O ₁	8	7	3	60
O ₂	3	8	9	70
O ₃	11	3	5	80
b _j	50	80	80	

					Row penalty
8	7	3	6	60	4 4
3	5	8	9	70	5 1
11	3	5	8	80	2 2
50	80	80			

5 ↑ 4 2
4 ↑ 2

Vogel's approximation method.

8	11	7	3	⑥
3	⑤	8	9	②
11	12	3	⑧	5
	-6	-3	0	

As number of allocations is $4 < 5$
Add ϵ , which is very small, to satisfy optimality condition.

for occupied cells

$$c_{ij} = u_i + v_j$$

for unoccupied cells.

$$d_{ij} = c_{ij} - (u_i + v_j)$$

but for all unoccupied cells, $d_{ij} > 0$

\therefore This itself is the optimal solution

	D_1	D_2	D_3
O_1	8	7	3
O_2	3	⑤	9
O_3	11	3	⑧

O_1 to $D_3 \Rightarrow 60$ units

O_2 to $D_1 \Rightarrow 50$ units & O_2 to $D_3 \Rightarrow 20$ units

O_3 to $D_2 \Rightarrow 80$ units

$$\text{Cost} = 60 \times 3 + 50 \times 3 + 20 \times 9 + 80 \times 3 = \underline{750}$$

SECTION-B

5. (a) Find a complete integral of $z(xp-yq)^2 = y^2 - x^2$

$$x \left(z \frac{\partial z}{\partial x} \right) - y \left(z \frac{\partial z}{\partial y} \right) = y^2 - x^2$$

$$\text{Let } z dz = dZ \Rightarrow \frac{z^2}{2} = Z$$

$$\Rightarrow x \left(\frac{\partial Z}{\partial x} \right) - y \left(\frac{\partial Z}{\partial y} \right) = y^2 - x^2$$

$$\therefore xP - yQ = y^2 - x^2$$

$$\Rightarrow xP + x^2 = yQ + y^2 = a \text{ (say)}$$

$$\Rightarrow P = \frac{a^2 - x^2}{x} ; Q = \frac{a^2 - y^2}{y}$$

$$\Rightarrow dZ = P dx + Q dy.$$

$$dZ = \left(\frac{a}{x} - x\right) dx + \left(\frac{a}{y} - y\right) dy$$

$$Z = a \log x - \frac{x^2}{2} + a \log y - \frac{y^2}{2} + b.$$

$$\Rightarrow \frac{z^2}{2} = a \log(xy) - \frac{1}{2}(x^2 + y^2) + b$$

$$z^2 = 2a \log(xy) - (x^2 + y^2) + b'$$

where $b' = 2b$.

(b) Solve (a) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy + \sin(2x+y)$ (10)

$$(D+D')(D-2D'+2)z = e^{2x+3y} + xy + \sin(2x+y)$$

$$CF = \phi_1(y-x) + e^{-2x} \phi_2(y+2x).$$

$$PI_1 = \frac{1}{(D+D')(D-2D'+2)} e^{2x+3y} = \frac{e^{2x+3y}}{(2+3)(2-6+2)}$$

$$= -\frac{1}{10} e^{2x+3y}$$

$$PI_2 = \frac{\sin(2x+y)}{D^2 - DD' - 2D'^2 + 2D + 2D'}$$

$$D^2 \rightarrow -4$$

$$D' \rightarrow -1$$

$$DD' \rightarrow -2$$

$$= \frac{1}{2(D+D')} \sin(2x+y)$$

$$= \frac{D-D'}{2(D^2-D'^2)} \sin(2x+y)$$

$$= -\frac{1}{6} [2 \cos(2x+y) - \cos(2x+y)]$$

$$= -\frac{1}{6} \cos(2x+y)$$

$$PI_3 = \frac{1}{(D+D')(D-2D'+2)} (xy)$$

$$= \frac{1}{2(D+D')} \left[1 + \frac{D-2D'}{2} \right]^{-1} (xy)$$

$$= \frac{1}{2(D+D')} \left[1 - \left(\frac{D-2D'}{2} \right) + \frac{1}{4} (-4DD') \right] (xy)$$

$$= \frac{1}{2(D+D')} \left[xy - \frac{1}{2}(y-2x) - 1 \right]$$

$$= \frac{1}{2D} \left[1 - \frac{D'}{D} + \left(\frac{D'}{D} \right)^2 - \dots \right] \left[xy - \frac{1}{2}(y-2x) - 1 \right]$$

$$= \frac{1}{4} x^2 y - \frac{1}{12} x^3 + \frac{1}{8} x^2 y - \frac{xy}{2} - \frac{x}{2}$$

$$z = \phi_1(y-x) + e^{-2x} \phi_2(y+2x) - \frac{1}{10} e^{2x+3y}$$

$$- \frac{1}{6} \cos(2x+3y) + \frac{1}{4} x^2 y - \frac{1}{12} x^3 + \frac{1}{8} x^2$$

(c) By using Newton forward interpolation formula, find the number of men getting wages between Rs. 10 and 15 from the following data.

Wages in Rs.	0-10	10-20	20-30	30-40
No. of Men	9	10	35	42

(10)

let y denote the no. of men getting wages less than x Rs

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
10	$y_0 = 9$	$\Delta y_0 = 10$	$\Delta^2 y_0 = 25$	$\Delta^3 y_0 = -18$
20	19			
30	54			
40	96			

$$x = 15; x_0 = 10; r = \frac{x - x_0}{h} = \frac{15 - 10}{10} = 0.5$$

$$\therefore y = y_0 + r \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0$$

$$= 9 + (0.5)(10) + \frac{r(r-1)}{2} \times 25 + \frac{r(r-1)(r-2)}{6} (-18)$$

$$= 9 + 5 + \left(-\frac{25}{8}\right) + \left(\frac{9}{8}\right) = 9.75$$

\therefore No. of men getting wages b/w Rs 10 & Rs 15

$$= 9.75 - 9$$

$$= 0.75$$

(d) Use Hamilton's equations to find the equation of motion of a particle of mass m moving on a straight line in simple harmonic motion.

$$T = \frac{1}{2} m \dot{x}^2 \quad ; \quad \text{Yes } F = ma = -\frac{m\mu x}{2} = -\frac{\partial V}{\partial x}$$

$$\Rightarrow V = + \frac{m\mu x^2}{2}$$

$$\therefore L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{m\mu x^2}{2}$$

$$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \Rightarrow \dot{x} = \frac{p_x}{m}$$

$$H = T + V = \frac{1}{2} m \dot{x}^2 + \frac{m\mu x^2}{2} = \frac{p_x^2}{2m} + \frac{m\mu x^2}{2}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\left[\frac{m\mu x}{2} \right] \quad \text{--- (1)}$$

$$\text{and } \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad \text{--- (2)}$$

$$\ddot{x} = \frac{\dot{p}_x}{m} = -\frac{\mu}{2} x$$

$$\therefore \ddot{x} = -\frac{\mu}{2} x$$

$$\therefore p_x = m \dot{x}$$

$$\dot{p}_x = m \ddot{x}$$

$$\therefore m \ddot{x} = -m\mu x$$

$\Rightarrow \ddot{x} = -\mu x$ is the equ of particle in S.H.M.

(c) Show that

$u = \frac{-2yz}{(x^2 + y^2)^2}$, $v = \frac{(x^2 - y^2)z}{(x^2 + y^2)^2}$, $w = \frac{y}{x^2 + y^2}$ are the velocity components of a possible liquid motion. Is this motion irrotational? (10)

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6. (a) Form a partial differential equation by eliminating the function

$$z = (x-y) \phi(x^2+y^2).$$

$$p = \frac{\partial z}{\partial x} = (x-y) \phi'(x^2+y^2) (2x).$$

$$q = \frac{\partial z}{\partial y} = (x-y) \phi'(x^2+y^2) (2y).$$

Multiply ① by y & ② by x &

$$(py - qx) = y \phi(x^2+y^2) + x \phi(x^2+y^2)$$

$$\Rightarrow \phi(x^2+y^2) = \frac{py - qx}{x+y} \quad \checkmark$$

$$\therefore z = (x-y) \cdot \frac{py - qx}{x+y}$$

$$\boxed{(x+y)z = (x-y)(py - qx)}$$

is the required P.D.E. \checkmark

(b) Find the characteristics of the equation $pq=z$, and determine the integral surface which passes through the parabola $x=0, y^2=z$. (15)

$$f = pq - z; \quad f_x = 0; \quad f_y = 0; \quad f_z = -1$$

$$f_p = q; \quad f_q = p.$$

$$\frac{dx}{dt} = f_p; \quad \frac{dy}{dt} = f_q; \quad \frac{dz}{dt} = pf_p + qf_q$$

$$\frac{dp}{dt} = -f_x - pf_z; \quad \frac{dq}{dt} = -f_y - qf_z.$$

$$\frac{dx}{dt} = q; \quad \frac{dy}{dt} = p; \quad \frac{dz}{dt} = 2pq$$

$$\frac{dp}{dt} = p; \quad \frac{dq}{dt} = q.$$

∴ initial values are

$$x_0 = 0; \quad y_0 = \lambda, \quad z_0 = \lambda^2$$

$$p_0 = \frac{\lambda}{2}; \quad q_0 = 2\lambda$$

$$\frac{dp}{p} = dt \Rightarrow p = \frac{\lambda}{2} e^t$$

$$q = 2\lambda e^t$$

$$dx = 2\lambda e^t dt \Rightarrow x = 2\lambda e^t - 2\lambda$$

$$x = 2\lambda(e^t - 1); \quad dy = \frac{\lambda}{2} e^t dt$$

$$y = \frac{\lambda}{2} e^t + \frac{\lambda}{2} \Rightarrow y = \frac{\lambda}{2} (e^t + 1)$$

$$\frac{dz}{dt} = 2\lambda^2 e^{2t}$$

$$z = \lambda^2 e^{2t}$$

$$\frac{x}{y} = \frac{2(e^t - 1)}{e^t + 1}$$

$$\Rightarrow x e^{t+x} = 4(y e^{t-y})$$

$$e^t (x - 4y) = -x - 4y$$

$$e^t = \frac{x + 4y}{4y - x}$$

$$x = 2\lambda \left(\frac{2x}{4y - x} \right) \Rightarrow \lambda = \frac{4y - x}{4}$$

Putting in z

$$z = \left(\frac{4y - x}{4} \right)^2 \left(\frac{x + 4y}{4y - x} \right)^2$$

$$\Rightarrow 16z = (x + 4y)^2$$

(c) Reduce $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form. $\Rightarrow x + x^2 t = 0$. (10)

$B^2 - 4AC < 0 \Rightarrow$ Elliptic.

characteristic equation $\Rightarrow \frac{dy}{dx} \pm ix = 0$.

$$\Rightarrow y + i\frac{x^2}{2} = c_1; \quad y - i\frac{x^2}{2} = c_2.$$

$$\text{let } u = y + i\frac{x^2}{2}; \quad v = y - i\frac{x^2}{2}.$$

choose $\xi = \phi(x, y) = y$; $\eta = \psi(x, y) = \frac{x^2}{2}$.

$$\phi_x = 0; \quad \phi_y = 1; \quad \phi_{xx} = 0; \quad \phi_{yy} = 0$$

$$\psi_x = x; \quad \psi_y = 0; \quad \psi_{xx} = 1; \quad \psi_{yy} = 0.$$

Given equation $a u_{xx} + b u_{xy} + c u_{yy} = 0$

where $a = 1$; $b = 0$; $c = x^2$ & $u = z$.

Canonical form is $A u_{\xi\xi} + B u_{\xi\eta} + C u_{\eta\eta} + R = 0$

where

$$A = 1 \cdot 0 + 0 + x^2 \cdot 1 = x^2; \quad B = 1 \cdot x^2 + 0 + 0 = x^2$$

$$C = 2 \cdot 1 \cdot 0 + 0 + 2 \cdot 0 = 0.$$

$$R = (1 \cdot 0 + 0 + x^2 \cdot 0) u_{\xi\xi} + (1 \cdot 1 + 0 + x^2 \cdot 0) u_{\eta\eta}.$$

$$= u_{\eta\eta}.$$

The canonical form.

$$x^2 u_{\xi\xi} + x^2 u_{\eta\eta} + u_{\eta\eta} = 0$$

where

$$u = z.$$

- d) Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions

$$u(0, y) = u(a, y) = 0 \text{ for } 0 \leq y \leq b$$

$$u(x, b) = 0 \text{ and } u(x, 0) = x(a-x) \text{ for } 0 \leq x \leq a.$$

The most feasible solution is

$$u(x, y) = (A \cos px + B \sin px)(C e^{py} + D e^{-py})$$

$$x=0 \rightarrow u=0.$$

$$\therefore \boxed{A=0}$$

$$x=a \rightarrow u=0$$

$$\therefore B \sin pa = 0 \Rightarrow \sin pa = 0 \Rightarrow pa = n\pi$$

$$\therefore \boxed{p = n\pi/a}$$

$$y=b \rightarrow u=0.$$

$$C e^{pb} + D e^{-pb} = 0 \Rightarrow C e^{pb} = -D e^{-pb}.$$

$$\Rightarrow \boxed{D = -C e^{2pb}}$$

$$u(x, y) = C e^{py} + D e^{-py}$$

$$= C [e^{py} - e^{-py} \cdot e^{2pb}]$$

$$= C e^{pb} [e^{p(y-b)} - e^{-p(y-b)}]$$

$$= 2C e^{pb} \sinh(p(y-b))$$

$$\therefore u(x, y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left[\frac{n\pi}{a}(y-b)\right] \quad \text{--- (1)}$$

where $b_n = 2BC e^{n\pi b/a}$

in (1) put $y=0$ & $u = x(a-x)$

$$x(a-x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)$$

where $B_n = b_n \sinh\left(-\frac{n\pi b}{a}\right)$

$$\therefore B_n = \frac{2}{a} \int_0^a x(a-x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a} \left[(ax-x^2) \left(-\frac{\cos kx}{k} \right) - (a-2x) \left(-\frac{\sin kx}{k^2} \right) + (-2) \left(\frac{\cos kx}{k^3} \right) \right]_0^a \quad \text{where } k = \frac{n\pi}{a}$$

$$= \frac{2}{a} \left[-\frac{2}{k^3} \cos(n\pi) + \frac{2}{k^3} \right]$$

$$B_n = \frac{4}{ak^3} [1 - \cos n\pi] = \begin{cases} \frac{8}{ak^3} & ; n \text{ is odd} \\ 0 & ; n \text{ is even} \end{cases}$$

$$u(x, y) = \frac{8a^2}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi}{a}(b-y)\right) \operatorname{cosech}\left(\frac{n\pi b}{a}\right)$$

Put $n = 2m - 1$

$$\Rightarrow u(x, y) = \frac{8a^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin\left[\frac{(2m-1)\pi}{a} x\right] \times \sinh\left[\frac{(2m-1)\pi}{a}(b-y)\right] \operatorname{cosech}\left[\frac{(2m-1)\pi b}{a}\right]$$

7. (a) Show that the equation $f(x) = 1 - xe^{1-x} = 0$ has a double root at $x=1$, find the root by using Newton-Raphson method. Also verify that the rate of convergence is linear. (12)

$$f(x) = 1 - xe^{1-x}$$

$$f'(x) = -[e^{1-x} + xe^{1-x}(-1)]$$

$$= -e^{1-x} [1-x]$$

As $f'(x)$ also has a root at $x=1$

$\Rightarrow f(x)$ has a ~~root~~ double root at $x=1$.

$$x_{n+1} = x_n - 2 \frac{f(x_n)}{f'(x_n)}$$

let $x_0 = 0$

$$x_{n+1} = x_n - 2 \left[\frac{1 - x_n e^{1-x_n}}{-e^{1-x_n} (1-x_n)} \right]$$

$$x_1 = 0.7357$$

$$x_2 = 0.97818$$

$$x_3 = 0.9998$$

$$\underline{x_4 = 0.99999 \approx 1.}$$

$\therefore x = 1$ is double root of $f(x)$.

$$e_0 = |x_1 - x_0| = 0.7357$$

$$e_1 = |x_2 - x_1| = 0.24248$$

$$e_2 = |x_3 - x_2| = 0.02162$$

clearly $\frac{e_1}{e_0} < 1$ but $\frac{e_2}{e_1} < 1$; $\frac{e_3}{e_2} < 1$

2) Linear rate of convergence.

(b) In a boolean Algebra B, for any 'a' and 'b' prove the following

$$a + \bar{a}b = a + b$$

(i) $(a+b)(a'+b')(a+b)(a'+b) = 0$

(ii) Compute $(3205)_{10}$ to the base 8.

(08)

$$\overline{a + \bar{a}b} = \bar{a} \cdot (a + \bar{b}) = \bar{a}a + \bar{a}\bar{b}$$

$$\overline{a + \bar{a}b} = \bar{a}\bar{b}$$

again taking complement

$$a + \bar{a}b = \overline{\bar{a}\bar{b}} = \bar{\bar{a}} + \bar{\bar{b}} = a + b$$

$$(a + \bar{b})(\bar{a} + \bar{b}) = (a\bar{a} + \bar{a}\bar{b} + a\bar{b} + \bar{b}\bar{b})$$

$$= (\bar{a}\bar{b} + a\bar{b} + \bar{b})$$

$$= \bar{a}\bar{b} + \bar{b} = \bar{b}(\bar{a} + 1) = \bar{b}$$

$$(a+b)(\bar{a}+b) = a\bar{a} + \bar{a}b + ab + b$$

$$= \bar{a}b + ab + b = \bar{a}b + b(a+1)$$

$$= \bar{a}b + b = b(\bar{a} + 1) = b$$

$$[(a + \bar{b})(\bar{a} + \bar{b})][a + b)(\bar{a} + b)]$$

$$= \bar{b} \cdot b$$

$$= 0$$

(iii) $(3205)_{10} = (6205)_8$

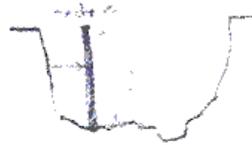
8	3205
8	400 - 5
8	50 - 0
	6 - 2

(c) A river is 80 ft wide. The depth d in feet at a distance x ft. from one bank is given by the following table:

x :	0	10	20	30	40	50	60	70	80
y :	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross section.

$$A = \int_{x=0}^{80} y \, dx.$$



using Simpson's $1/3$ rd rule.

$$h = 10;$$

$$A = \frac{h}{3} \left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6) \right]$$

$$= \frac{10}{3} \left[(0 + 3) + 4(4 + 9 + 15 + 8) + 2(7 + 12 + 14) \right]$$

$$= \underline{\underline{710 \text{ sq ft.}}}$$

(d) Design an algorithm for Runge-Kutta method.

step 1: enter x_0 , y_0 , $f(x, y)$, step size h , y_0 .

step 2: enter n , no. of iterations.

step 3: for $i = 1$ to n , step 1.

step 4: $R_1 = h f(x_0, y_0)$

$$R_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{R_1}{2}\right)$$

$$R_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{R_2}{2}\right)$$

$$R_4 = h f(x_0 + h, y_0 + R_3)$$

step 5: $y = y_0 + \frac{1}{6} (R_1 + 2R_2 + 2R_3 + R_4)$

step 6: $x_0 = x_0 + h$.

$$y_0 = y.$$

step 7: end for.

step 8: Print x_0, y_0 .