

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-I(M) IAS / T- 05

# MATHEMATICS

by K. VENKANNA

The person with 14 years of Teaching Experience

## FULL TEST P-I

Time: Three Hours

Maximum Marks: 250

### INSTRUCTIONS

1. This question paper-cum-answer booklet has 51 pages and has 33 PART / SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/tacksons carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name

Gowtham Potru

Roll No.

014-773

Test Centre

ORN

Medium

English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Gowtham

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

### IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

**DO NOT WRITE ON  
THIS SPACE**

200 | 250

### INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			08
	(b)			1
	(c)			8
	(d)			8
	(e)			8
2	(a)	*		7
	(b)			8
	(c)			13
	(d)			12
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			5
	(b)			8
	(c)			7
	(d)			8
	(e)			7
6	(a)			8
	(b)			11
	(c)			11
	(d)			10
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			10
	(b)			6
	(c)			10
	(d)			16
<b>Total Marks</b>				<b>200</b>

**DO NOT WRITE ON  
THIS SPACE**

## SECTION-A

1. (a) Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into:  $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$  and making a clever choice of  $\lambda$ . (10)

Given that characteristic polynomial  $|A - \lambda I|$  can be factored into  $(\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$  where  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  are eigenvalues of matrix 'A'.

$$\therefore |A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda) \rightarrow ①$$

lets choose  $\lambda = 0$

then ① becomes

$$|A - (0)I| = (\lambda_1 - 0)(\lambda_2 - 0) \dots (\lambda_n - 0)$$

$$08 \rightarrow |A - 0I| = (\lambda_1)(\lambda_2) \dots (\lambda_n)$$

$$\Rightarrow |A| = \lambda_1 \lambda_2 \dots \lambda_n$$

$\therefore \det(A)$  is equal to the product of eigenvalues

1. (b) If  $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ , then  $\det(A - \lambda I)$  is  $(\lambda - a)(\lambda - d)$ . Check the Cayley-Hamilton statement that  $(A - aI)(A - dI) = \text{zero matrix}$ . (10)

$$\text{Given } A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

$$\begin{aligned} A - aI &= \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} - a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{find } |A - aI| \\ &= \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix}. \quad \text{not invertible} \end{aligned}$$

$$\begin{aligned} A - dI &= \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} - d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\therefore (A - aI)(A - dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix} \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \text{zero matrix} \rightarrow ①$$

$$\text{Given that } |A - \lambda I| = (\lambda - a)(\lambda - d) \rightarrow ②$$

Cayley-Hamilton theorem :

Every square matrix satisfies its characteristic polynomial.

∴ from ① & ② we can say that Cayley-Hamilton statement is proven for A

1(c). Show that  $\int_0^{\pi/2} \log(\sin x) dx$  is convergent and hence evaluate it.

(10)

$$\text{Let } I = \int_0^{\pi/2} \log(\sin nx) dx \rightarrow ①$$

$$\Rightarrow I = \int_0^{\pi/2} \log(\sin(\frac{\pi}{2}-x)) dx$$

$$\Rightarrow I = \int_0^{\pi/2} \log(\cos x) dx \rightarrow ②$$

Adding ① & ②

$$2I = \int_0^{\pi/2} \log(\sin nx \cos n) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} [\log(\frac{1}{2}) + \log(\sin 2x)] dx$$

$$\Rightarrow 2I = -\frac{\pi}{2} \log 2 + \int_0^{\pi/2} \log(\sin 2x) dx \rightarrow ③$$

Now,  $\int_0^{\pi/2} \log(\sin 2x) dx$  put  $2x = t$   
 $dx = \frac{1}{2} dt$

$$= \frac{1}{2} \int_0^{\pi} \log(\sin t) dt$$

$$= \int_0^{\pi} \log(\sin t) dt \quad [ \because \sin t \text{ is symmetrical in } [0, \pi] ]$$

$$= I.$$

$$\therefore 2I = -\frac{\pi}{2} \log 2 + I$$

$$\Rightarrow \int_0^{\pi/2} \log(\sin x) dx = -\frac{\pi}{2} \log 2$$

- I. (d) The plane  $lx + my = 0$  is rotated about the line of intersection with the plane  $z = 0$  through an angle  $\alpha$ . Prove that the equation to the plane in its new position is (10)

$$lx + my \pm z\sqrt{l^2 + m^2} \tan \alpha = 0$$

The given plane  $lx + my = 0 \rightarrow \textcircled{1}$

Let the plane that passes through intersection

of  $\textcircled{1}$  with the plane  $z=0$  be

$$lx + my + \lambda z = 0 \rightarrow \textcircled{2}$$

Given that plane  $\textcircled{2}$  is obtained by rotating plane  $\textcircled{1}$  through an angle  $\alpha$

$\Rightarrow$  normals to the planes  $\textcircled{1}$  &  $\textcircled{2}$  are at an angle ' $\alpha$ ' to each other

$$\Rightarrow \frac{(\ell)(\ell) + (m)(m) + (\lambda)(\alpha)}{\sqrt{\ell^2+m^2} \sqrt{\ell^2+m^2+\lambda^2}} = \sec \alpha$$

$$\Rightarrow \sec^2 \alpha = \frac{\ell^2+m^2+\lambda^2}{\ell^2+m^2} \Rightarrow \tan \alpha = \frac{\lambda^2}{\ell^2+m^2}$$

$$\Rightarrow \lambda = \pm \sqrt{\ell^2+m^2} \tan \alpha.$$

$\therefore$  The required plane equation is

Q8  $\ell x + m y \pm z \sqrt{\ell^2+m^2} \tan \alpha = 0$

1. (e) A point moves so that the sum of the squares of its distances from the six faces of a cube is constant, show that its locus is a sphere. (10)

Let the point be  $P(\alpha, \beta, \gamma)$

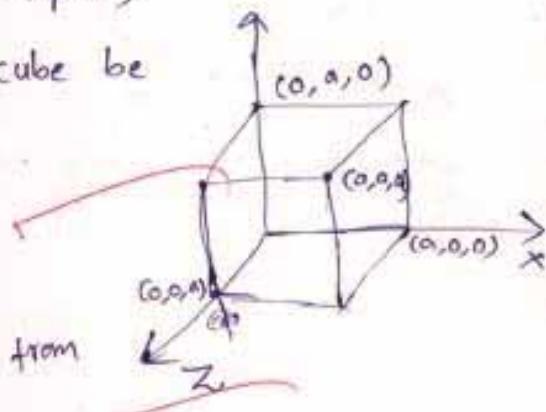
and the faces of a cube be

$$x=0; x=a; y=0; y=a;$$

$$z=0; z=a$$

Distance of  $P(\alpha, \beta, \gamma)$  from  
six faces will be

$$|\alpha|, |\alpha-a|, |\beta|, |\beta-a|, |\gamma|, |\gamma-a|$$



Q8

Given that sum of squares of these distances  
is constant (say  $k^2$ )

$$\therefore \alpha^2 + (\alpha-a)^2 + \beta^2 + (\beta-a)^2 + \gamma^2 + (\gamma-a)^2 = k^2$$

$$\Rightarrow 2(\alpha^2 + \beta^2 + \gamma^2) - 2a(\alpha + \beta + \gamma) + 3a^2 - k^2 = 0$$

$\therefore$  The locus of the point  $P(x, y, z)$  is

$$2(x^2 + y^2 + z^2) - 2a(x + y + z) + 3a^2 - k^2 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - ax - ay - az + \frac{3a^2 - k^2}{2} = 0$$

The above equation is a sphere with  
centre  $(\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$

2. (a) Find the rank and the nullspace of

(10)

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

(i) for the matrix  $\underline{A}$

$$\text{Given } A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3 \quad R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \underline{\text{rank}(A) = 2}$$

to find null space  $\rightarrow$  let  $AX = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow z = 0; y = 0; x + y + z = 0$$

$$\Rightarrow z=0; y=-x$$

$\therefore$  null space of A is  $\{(+x, -x, 0) \mid x \in \mathbb{R}\}$   
 $\Rightarrow \text{null space}(A) = \{(1, -1, 0)\}$

$$\therefore \text{null space}(A) = \{(1, -1, 0)\}$$

(ii) for the matrix B

Given  $B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$   
 $R_3 \rightarrow R_3 - R_2$

$$\therefore \text{Rank}(B) = 2$$

Let  $Bx = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow z+2w=0; x+y+z=0$$

$$\left. \begin{array}{l} x+y+z=0 \\ z+2w=0 \end{array} \right\} \Rightarrow$$

The above system has 2 independent variables ( $y, w$ )

Let  $y=0, w=1 \Rightarrow z=-2; x=2$

Let  $y=1, w=0 \Rightarrow z=0; x=-1$

$\therefore \text{null space}(B) = \{(2, 0, -2, 1), (-1, 1, 0, 0)\}$

2. (b)  $M$  is any 2 by 2 matrix and  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ . The linear transformation  $T$  is defined by  $T(M) = AM$ . What rules of matrix multiplication show that  $T$  is linear? (10)

Given linear transformation 15

$$T(M) = AM.$$

Let  $M, N$  be two matrices  $(2 \times 2)$  and  $\alpha, \beta$  be real.

$$\text{Now, } T(\alpha M + \beta N) = A(\alpha M + \beta N)$$

$$= A(\alpha M) + A(\beta N)$$

$$= \alpha(AM) + \beta(AN)$$

$$= \alpha T(M) + \beta T(N).$$

Q8

$\therefore T(M)$  is linear.

(i) In the above proof we used

$$A(\alpha M + \beta N) = A(\alpha M) + A(\beta N)$$

It is obtained by using distributive property of matrix multiplication

$$\underline{A(B+C) = AB+AC}$$

(ii) Also, we used

$$A(\alpha M) = \alpha(AM)$$

It is obtained by using association of scalar multiplication of matrix multiplication

$$\underline{A(kB) = k(AB)}$$

2. (c) Compute  $f_{xy}(0,0)$  and  $f_{yx}(0,0)$  for the function

(15)

$$f(x,y) = \begin{cases} (x^2 + y^2) \log(x^2 + y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Also discuss the continuity of  $f_x$  and  $f_y$  at  $(0,0)$

~~$f(x,y)$~~

$$f(x,y) = (x^2 + y^2) \log(x^2 + y^2)$$

$$\Rightarrow f_x = \frac{\partial f}{\partial x} = 2x \log(x^2 + y^2) + \frac{x^2 + y^2}{(x^2 + y^2)} (2x)$$

$$= 2x \log(x^2 + y^2) + 2x$$

$$\Rightarrow f_{xy} = \frac{\partial f_x}{\partial y} = \frac{2x(2y)}{x^2 + y^2} = \frac{4xy}{x^2 + y^2} \rightarrow ①$$

Similarly

$$f_{yx} = \frac{4xy}{x^2 + y^2} \rightarrow ②$$

$$\begin{aligned} f_{xy}(0,0) &= \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0,0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 \log h^2 - 0}{h} = \lim_{h \rightarrow 0} h \log h \\ &= \lim_{h \rightarrow 0} \frac{\log h}{\frac{1}{h}} = -2 \lim_{h \rightarrow 0} \frac{1}{h} = 0 \end{aligned}$$

$$\Rightarrow f_{xy}(0,0) = 0$$

$$f_{yx}(0,0) = \lim_{k \rightarrow 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \rightarrow 0} \frac{k^2 \log k^2 - 0}{k}$$

$$= 0$$

$$\Rightarrow f_{yx}(0,0) = 0$$

continuity of  $f_{xy} = \frac{4xy}{x^2+y^2} = f_{yx}$

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{4xy}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{4mx^2}{m^2x^2 + x^2} = \frac{4m}{1+m^2}$$

along  $y=mx$

~~limiting~~ to which is different for different values of  $m$

$\therefore f_{xy}, f_{yx}$  are not continuous at  $(0,0)$

2. (d) The sections of the enveloping cone of the surface  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  whose vertex is  $P(x_1, y_1, z_1)$  by the plane  $z = 0$  is a circle. Find the locus of the vertex  $P$ .

(15)

Given  $P(x_1, y_1, z_1) \rightarrow ①$

and the surface  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \rightarrow ②$

Enveloping cone of ② with vertex as P

" SS<sub>1</sub> = T<sup>2</sup>"

$$\begin{aligned} & \Rightarrow \left( \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} \right) \left( \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) \\ & = \left( \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right)^2 \\ & \rightarrow ③ \end{aligned}$$

The section of the above cone by the plane  $z=0$

is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = \left(\frac{xy_1}{a^2} + \frac{yz_1}{b^2} - 1\right)^2$$

Given that above equation represents circle

$\Rightarrow$  coef  $(xy)$  = 0 & coef  $x^2, y^2$  are same

$$\Rightarrow 2\left(\frac{y_1}{a^2}\right)\left(\frac{y_1}{b^2}\right) = 0 \Rightarrow y_1 = 0 \quad \rightarrow \textcircled{4}$$

$$\text{and } \frac{1}{a^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) - \frac{y_1^2}{a^4} = \frac{1}{b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) - \frac{y_1^2}{b^4}$$

$$\Rightarrow \frac{y_1^2}{a^4 b^2} + \frac{z_1^2}{a^2 c^2} - \frac{1}{a^2} = \frac{y_1^2}{a^2 b^2} + \frac{z_1^2}{b^2 c^2} - \frac{1}{b^2} \rightarrow \textcircled{5}$$

from \textcircled{4} if  $y_1 = 0$

then \textcircled{5} becomes  $\frac{y_1^2}{a^2 b^2} + \frac{z_1^2}{a^2 c^2} - \frac{1}{a^2} = \frac{z_1^2}{b^2 c^2} - \frac{1}{b^2}$

$$\Rightarrow c^2 y_1^2 + (b^2 - a^2) z_1^2 = (b^2 - a^2) c^2$$

$$\Rightarrow \frac{y_1^2}{b^2 - a^2} + \frac{z_1^2}{c^2} = 1$$

... locus of 'P' is  $\frac{y^2}{b^2 - a^2} + \frac{z^2}{c^2} = 1 ; y=0$

If  $y_1 = 0$

\textcircled{5} becomes

$$b^2 z_1^2 - b^2 c^2 = c^2 y_1^2 + a^2 z_1^2 - a^2 c^2$$

$$\Rightarrow \frac{y_1^2}{a^2 - b^2} + \frac{z_1^2}{c^2} = 1$$

... locus of 'P' is  $\frac{y^2}{a^2 - b^2} + \frac{z^2}{c^2} = 1 ; y=0$

$\frac{y^2}{a^2 - b^2} + \frac{z^2}{c^2} = 1 ; y=0$

3. (a) (i) Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

(ii) Do the vectors  $(1, 1, 3)$ ,  $(2, 3, 6)$  and  $(1, 4, 3)$  form a basis for  $\mathbb{R}^3$ ? (20)

IMS-Institute Of Mathematical Sciences

IMS-Institute Of Mathematical Sciences

3. (b) Find the shortest distance from the origin to the hyperbola  $x^2 + 8xy + 7y^2 = 225$ ,  $z=0$ . (15)

IMS-Institute Of Mathematical Sciences

3. (c) Show that the plane  $x+y-z=0$  cuts the coneoid  $4x^2+2y^2+z^2+3yz+zx-1=0$  in a circle. Find also the radius of this circle. (15)

5)

P.T.O.

P.T.O.

IMS-Institute Of Mathematical Sciences

4. (a) Let  $V$  and  $W$  be the following subspaces of  $\mathbb{R}^4$ :  $V = \{(a, b, c, d) / b - 2c + d = 0\}$ ,  
 $W = \{(a, b, c, d) / a = d, b = 2c\}$ . Find bases and dimensions of  $V$ ,  $W$  and  $V \cap W$ . Hence prove that  
 $\mathbb{R}^4 = V + W$ .

(15)

IMS-Institute Of Mathematical Sciences

4. (b) Find the volume cut off the sphere  $x^2 + y^2 + z^2 = a^2$  by the cone  $x^2 + y^2 = z^2$ .

(15)

IMS-Institute Of Mathematical Sciences

4. (c) Prove that the tangent planes to the hyperboloid  $\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) - \left(\frac{z^2}{c^2}\right) = 1$  which are parallel to tangent planes to the cone  $\frac{b^2 c^2 x^2}{c^2 - b^2} + \frac{c^2 a^2 y^2}{c^2 - a^2} + \frac{a^2 b^2 z^2}{a^2 + b^2} = 0$  cut the surface in perpendicular generators. (20)

IMS-Institute Of Mathematical Sciences

IMS-Institute Of Mathematical Sciences

## SECTION-B

5. (a) Prove that  $1/(x+y+1)^4$  is an integrating factor of  $(2xy - y^2 - y)dx + (2xy - x^2 - x)dy = 0$ , and find the solution of this equation. (10)

Multiplying  $\frac{1}{(x+y+1)^4}$  with the given equation

$$\frac{2xy - y^2 - y}{(x+y+1)^4} dx + \frac{2xy - x^2 - x}{(x+y+1)^4} dy = 0 \rightarrow ①$$

If the above  $Mdx + Ndy$  is exact

$$\text{then } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 0$$

$$\frac{\partial M}{\partial y} = \frac{2x - 2y - 1}{(x+y+1)^4} - \frac{4(2xy - y^2 - y)}{(x+y+1)^5} \rightarrow \frac{2y - 2x - 1}{(x+y+1)^4} + \frac{4(2xy - x^2 - x)}{(x+y+1)^5} = 0$$

$$= \frac{4(x-y)}{(x+y+1)^4} + \frac{4(y^2 + y - x^2 - x)}{(x+y+1)^5}$$

$$= \frac{4[(x-y)(x+y+1) + (y^2 + y - x^2 - x)]}{(x+y+1)^5}$$

$$= 4 \frac{[x^2 + xy + x - x^2 - y^2 - y + y^2 + y - x^2 - x]}{(x+y+1)^5} = 0$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow ① \text{ is exact}$$

$\Rightarrow \frac{1}{(x+y+1)^4}$  is I.F. of the given eqn.

Solution of ① is  $\int(M) dx \underset{(y=\text{const})}{=} C + \int(N) dy \underset{(\text{ind. of } x)}{=} C$

$$\Rightarrow \int \frac{2xy - y^2 - y}{(x+y+1)^4} dx = C$$

$$\Rightarrow \frac{y^3 + y - 2xy}{3(x+y+1)^3} - \frac{x}{3(x+y+1)^2} = C$$

$$\Rightarrow xy + C(x+y+1)^3 = 0$$

5. (b) Examine for singular solution and extraneous loci,  $y + px = x^4 p^2$

(10)

$$y + pn = x^4 p^2 \rightarrow ①$$

$$\Rightarrow p^2 x^4 - pn - y = 0$$

p-discriminate

$$\text{is } "B^2 - 4AC"$$

$$= x^2 + 4(y)(x^4) = x^2 + 4x^4 y \\ = x^2(1 + 4x^2 y)$$

$$= ET^2 C$$

~~solve for  $x$~~ 

$$y + pn \pm p^2 n^4 \rightarrow ①$$

partially differentiate ~~w.r.t.~~  $x$

$$\Rightarrow p + p + n \frac{dp}{dn} = 4x^2 p \frac{dp}{dn} + p^2 4n^3$$

$$\Rightarrow (2p + n \frac{dp}{dn}) = 2p n^3 [ap + n \frac{dp}{dn}]$$

$$\begin{aligned} & \text{separate } \frac{dp}{dn} \text{ on one side} \\ & \Rightarrow \frac{dp}{dn} = \frac{2p n^3}{2p + n} \frac{1}{2n^3} \\ & \Rightarrow \frac{dp}{p} = \frac{1}{2n^3} dn \\ & \Rightarrow p = \frac{1}{2n^3} dn + C \\ & \Rightarrow y = -\frac{1}{4x^2} \dots + C \\ & \Rightarrow 1 + 4p^2 y = \frac{1}{4n^2} C^2 \end{aligned}$$

$$p = \frac{1}{2n^3} \Rightarrow \text{from } ① \quad 4x^2 y + 1 = 0$$

singular soln

$$\frac{\partial P}{\partial n} + n \frac{\partial^2 P}{\partial n^2} = 0 \Rightarrow \frac{\partial^2 P}{\partial n^2} + \frac{1}{n} \frac{\partial P}{\partial n} = 0$$

$$\Rightarrow \log P + \log n^2 = \log c^2$$

$$\Rightarrow P n^2 = c^2 \Rightarrow P = \frac{c^2}{n^2}$$

From ①  $y + \frac{c^2}{n^2} = n^4 \cdot \frac{c^2}{n^4}$

$$\Rightarrow c^2 n - c - n^2 y = 0$$

c - discriminant

$$= \frac{1 + 4n^2y}{B^2 - 4AC} = EN^2c^3$$

$$\therefore EN^2c = (1 + 4n^2y)$$

$$\therefore \text{singular soln } 'E' \quad 1 + 4n^2y$$

Tac locus

Node Locus

Cusp Locus

$c = 0$

$$T = y$$

$$N = 0$$

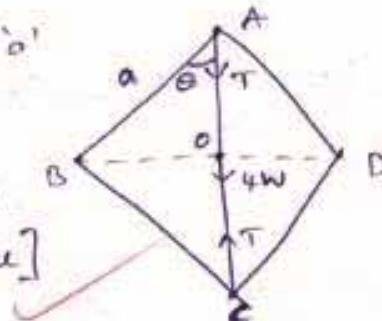
$$C = 0$$

✓

5. (c) Four uniform rods are freely jointed at their extremities and form a parallelogram  $ABCD$ , which is suspended by the joint  $A$ , and is kept in shape by a string  $AC$ . Prove that the tension of the string is equal to half the weight of all the four rods. (10)

Let the length of the rod is 'a'  
and weight 'w'

The total weight  $4w$  acts  
at 'o' [centre of the figure]



Let  $\angle BAO = \theta$

$$\Rightarrow AO = a \cos \theta ; AC = 2a \cos \theta$$

So we produce a little displacement in the figure

using virtual work

$$-T\delta(\Delta C) + 4W\delta(\Delta \theta) = 0$$

$$\Rightarrow -T\delta(2a\cos\theta) + 4W\delta(a\cos\theta) = 0$$

$$\Rightarrow (2aT\sin\theta - 4aW\sin\theta)\delta\theta = 0$$

~~$$\Rightarrow 2aT\sin\theta = 4aW\sin\theta$$~~

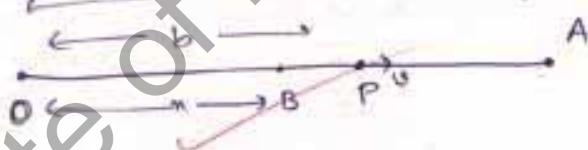
~~$$\Rightarrow T = 2W$$~~

~~$$\Rightarrow T = \frac{1}{2}(4W)$$~~

$\therefore$  tension is equal to half of the weight  
of all four rods

5. (d) A particle is performing a simple harmonic motion of period  $T$  about centre  $O$  and it passes through a point  $P$  where  $OP = b$  with velocity  $v$  in direction  $OP$ , prove that the time which elapses before it returns to  $P$  is:

$$\frac{T}{\pi} \tan^{-1}\left(\frac{vT}{2\pi b}\right) \quad (10)$$



$$\frac{d^2x}{dt^2} = -\mu x \quad [\text{SHM}]$$

$$\Rightarrow \frac{d^2x}{dt^2} \frac{dx}{dt} = -\mu x^2 \frac{dx}{dt}$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = -\mu x^2 + C$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2) ; \Rightarrow v^2 = \mu(a^2 - b^2) \rightarrow (1)$$

$$\Rightarrow \frac{dx}{dt} = -\sqrt{\mu(a^2 - x^2)}$$

$$\Rightarrow dt = -\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \frac{dx}{\sqrt{a^2-x^2}}$$

$$\Rightarrow t_{PO} = \int_b^0 -\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \frac{dx}{\sqrt{a^2-x^2}} = \left[ \frac{-1}{\sqrt{1-\frac{x^2}{a^2}}} \cos^{-1} \frac{x}{a} \right]_b^0$$

$$\Rightarrow t_{PO} = \frac{1}{\sqrt{1-\frac{b^2}{a^2}}} \left[ \frac{\pi}{2} - \cos^{-1} \frac{b}{a} \right]$$

Now,  ~~$t_{PA}$~~   $t_{PA} = t_{AP}$   
and  $t_{AP} = t_{AO} - t_{PO} = \frac{T}{2} - t_{PO}$

$$\therefore 2t_{PA} = 2 - \frac{1}{\sqrt{1-\frac{b^2}{a^2}}} \left( \pi - 2 \cos^{-1} \frac{b}{a} \right); \text{ we know } T = \frac{2\pi}{\sqrt{1-\frac{b^2}{a^2}}}$$

From ①  $\omega^2 = \mu(a^2 - b^2)$

~~$t_{PA}$~~  on simplification,  ~~$t_{PA} = \frac{T}{\pi} \tan^{-1} \left( \frac{bT}{2\pi a} \right)$~~

5. (e) If  $f = \nabla(\vec{a} \cdot \nabla r^{-1})$ , show that  $\operatorname{div} f = 0$ , and  $f = \operatorname{curl} g$ , where  $g = -\vec{a} \times \nabla(r^{-1})$ . (10)

$$\vec{a} \cdot \nabla \left( \frac{1}{r} \right) = \vec{a} \cdot \frac{-1}{r^2} \left( \frac{\vec{r}}{r} \right)$$

$$= -\frac{(\vec{a} \cdot \vec{r})}{r^3} = -\frac{(a_1 x + a_2 y + a_3 z)}{r^3} = h$$

$$\therefore f = \nabla \left[ -\left( \frac{\vec{a} \cdot \vec{r}}{r^3} \right) \right] = \nabla(h)$$

$$\frac{\partial h}{\partial x} = -\frac{a_1}{r^3} + \frac{(a_1 x + a_2 y + a_3 z)}{r^5} \cdot \left( \frac{3}{r^2} \right) \lambda^2 x$$

$$\therefore f = \frac{\partial h}{\partial x} \hat{i} + \frac{\partial h}{\partial y} \hat{j} + \frac{\partial h}{\partial z} \hat{k}$$

$$= -\frac{\vec{a}}{r^3} + \frac{3(\vec{a} \cdot \vec{r})}{r^5} \vec{r}$$

$$\begin{aligned}
 \text{Now, } \nabla \cdot \vec{A} &= \nabla \cdot \left[ -\frac{\vec{a}}{r^3} \right] + \nabla \cdot \left[ 3 \frac{(\vec{a} \cdot \vec{r})}{r^5} \cdot \vec{r} \right] \\
 &= \nabla \left( \frac{1}{r^3} \right) \cdot (-\vec{a}) + \frac{1}{r^3} (\nabla \cdot \vec{a}) + 3 \nabla \left( \frac{\vec{a} \cdot \vec{r}}{r^5} \right) \cdot \vec{r} + 3 \frac{(\vec{a} \cdot \vec{r})}{r^5} (\nabla \cdot \vec{r}) \\
 &= 0 \quad (\text{on simplification})
 \end{aligned}$$

Given  $\vec{g} = -\vec{a} \times \nabla \left( \frac{1}{r} \right)$

$$\begin{aligned}
 \text{We know } \nabla \times (\vec{A} \times \vec{B}) &= (\nabla \cdot \vec{B}) \vec{A} + (\vec{B} \cdot \nabla) \vec{A} \\
 &\quad - (\nabla \cdot \vec{A}) \vec{B} + (\vec{A} \cdot \nabla) \vec{B}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \nabla \times (\vec{g}) &= \nabla \times \left( -\vec{a} \times \nabla \left( \frac{1}{r} \right) \right) \\
 &= \cancel{\nabla \cdot \left[ \nabla \left( \frac{1}{r} \right) \right]} (-\vec{a}) + (\nabla \left( \frac{1}{r} \right) \cdot \nabla) (-\vec{a}) \\
 &\quad - (\nabla \cdot (-\vec{a})) \nabla \left( \frac{1}{r} \right) + (-\vec{a} \cdot \nabla) \nabla \left( \frac{1}{r} \right) \\
 &= \nabla (\vec{a} \cdot \nabla \frac{1}{r}) \quad \cancel{=} \\
 &\quad (\text{on simplification})
 \end{aligned}$$

*( $\because \vec{a}$  is constant)*

(10)

6. (a) Prove that the orthogonal trajectories of the curves  $A = r^2 \cos \theta$  are the curves  $B = r \sin^2 \theta$ .

(10)

Given curves  $r^2 \cos \theta = A \rightarrow ①$

differentiating ① w.r.t.  $\theta$

 $= h$ 

$$\cancel{-r^2 \sin \theta} + 2r \cos \theta \frac{dr}{d\theta} = 0 \rightarrow ②$$

To find orthogonal trajectories of ① replace

$\frac{dr}{d\theta}$  in ② by  $\underline{-r^2 \frac{d\theta}{dr}}$

$\therefore$  ② becomes

$$-r^2 \sin \theta + 2r \cos \theta (-r^2) \frac{d\theta}{dr} = 0$$

$$\Rightarrow \cancel{-r^2 \sin \theta + 2r^2 \cos \theta} \frac{d\theta}{dr} = 0$$

$$\Rightarrow \frac{dh}{\lambda} + \frac{2 \cos \theta}{\sin \theta} d\theta = 0$$

integrating

$$\Rightarrow \int \frac{dh}{\lambda} + \int \frac{2 \cos \theta}{\sin \theta} d\theta = \Theta C$$

$$\Rightarrow \log h + 2 \log \sin \theta = C$$

$$\Rightarrow \log h + 2 \log \sin^2 \theta = \log B \quad [ \text{where } C = \log B ]$$

$$\Rightarrow \underline{\lambda \sin^2 \theta = B} \quad [B \rightarrow \text{constant}]$$

108'

6. (b) Solve the differential equation  $(x^3 D^3 + 3x^2 D^2 + xD + 1)y = x \log x$  (13)

$$\text{Let } x = e^z \Rightarrow z = \log x$$

$$\text{and } xD = D_1, \quad \text{where } D = \frac{d}{dx}; \quad D_1 = \frac{d}{dz}$$

$$x^2 D^2 = D_1(D_1 - 1)$$

$$x^3 D^3 = D_1(D_1 - 1)(D_1 - 2)$$

Given eqn becomes

$$(D_1^3 - 3D_1^2 + 2D_1 + 3D_1^2 - 3D_1 + D_1 + 1)y = ze^z$$

$$\Rightarrow (D_1^3 + 1)y = ze^z$$

Complementary Function

$$D_1^3 + 1 = 0 \Rightarrow D_1 = -1; \frac{1}{2} + i\frac{\sqrt{3}}{2}; \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\therefore y = c_1 e^{-z} + e^{V_2 z} [c_2 \cosh \frac{\sqrt{3}}{2} z + c_3 \sinh \frac{\sqrt{3}}{2} z]$$

$$= \frac{c_1}{z} + \sqrt{z} \left[ c_2 \cosh \left( \frac{\sqrt{3}}{2} \log z \right) + c_3 \sinh \left( \frac{\sqrt{3}}{2} \log z \right) \right]$$

Particular integral

$$\begin{aligned} & = \frac{1}{D_1^3 + 1} (ze^z) \\ & = e^z \frac{1}{(D_1 + 1)^3 + 1} (z) \\ & = e^z \frac{1}{D_1^3 + 3D_1^2 + 3D_1 + 2} (z) \\ & = \frac{e^z}{2} \left[ 1 + \frac{D_1^3 + 3D_1^2 + 3D_1}{2} \right]^{-1} (z) \\ & = \frac{e^z}{2} \left[ 1 + \frac{3D_1}{2} + \dots \right] (z) \\ & = \frac{e^z}{2} \left[ z - \frac{3}{2} \right] \\ & = \frac{1}{2} [ze^z - \frac{3}{2}e^z] \\ & = \frac{1}{2} [z \log z - \frac{3}{2}z] \end{aligned}$$

~~A~~ solution of the given equation is

$$y = \frac{c_1}{z} + \sqrt{z} \left[ c_2 \cosh \left( \frac{\sqrt{3}}{2} \log z \right) + c_3 \sinh \left( \frac{\sqrt{3}}{2} \log z \right) \right] + \frac{1}{2} \left( \log z - \frac{3}{2} \right)$$

6. (c) Apply the method of variation of parameters to solve  $y_1 + a^2 y = \cot ax$

(13)

Given equation  $(D^2 + a^2) y = \cot ax \rightarrow (1)$

$$\text{C.F} \quad D^2 + a^2 = 0 \Rightarrow D = \pm ia$$

$$\therefore y = c_1 \cos ax + c_2 \sin ax$$

$$\text{let } u = \cos ax; v = \sin ax.$$

If  $y = A u + B v$  is solution of (1)

then by Method of Variation of parameters

$$A = - \int \frac{v R}{w} dx + C_1; \quad B = \int \frac{u R}{w} dx + C_2$$

$$\text{where } R = \cot ax.$$

$$w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = 'a'$$

$$\therefore A = - \int \frac{\sin ax \cot ax}{a} dx + C_1$$

$$= - \int \frac{\cos ax}{a} dx + C_1$$

$$= - \frac{\sin ax}{a} + C_1$$

$$\text{and } B = \int \frac{\cos ax \cot ax}{a} dx + C_2$$

$$= \frac{1}{a} \int \frac{\cot^2 ax}{\sin ax} dx + C_2$$

$$= \frac{1}{a} \int \frac{1 - \sin^2 ax}{\sin ax} dx + C_2$$

(13)

$$\begin{aligned}
 &= \frac{1}{\alpha} \int (\cot \alpha x - \tan \alpha x) dx + C_1 \\
 &= \frac{1}{\alpha^2} [\log(\cot \alpha x - \tan \alpha x) + \cot \alpha x] + C_2
 \end{aligned}$$

$\therefore$  solution of ① is

$$y = \left( -\frac{\tan \alpha x}{\alpha^2} + C_1 \right) \cot \alpha x + \left( \frac{1}{\alpha^2} \log(\cot \alpha x - \tan \alpha x) + \cot \alpha x \right) \tan \alpha x + C_2 \tan \alpha x$$

$$\Rightarrow y = C_1 \cot \alpha x + C_2 \tan \alpha x + \frac{\tan \alpha x}{\alpha^2} \log(\cot \alpha x - \tan \alpha x)$$

6. (d) By using Laplace transform method solve the differential equation

$$(D^2 - D - 2)y = 20 \sin 2x, \text{ subject to initial conditions } y = -1, Dy = 2 \text{ when } t = 0.$$

(14)

Given D.E  $\therefore y(0) = -1 ; y'(0) = 2$

$$y'' - y' - 2y = 20 \sin 2t \rightarrow ①$$

applying Laplace transform on both sides

$$\Rightarrow L\{y''\} - L\{y'\} - sL\{y\} = 20L\{\sin 2t\}$$

$$\Rightarrow s^2 L\{y\} - s y(0) - y'(0) - sL\{y\} + y(0) - 2L\{y\} = \frac{20(2)}{s^2 + 4}$$

$$\Rightarrow (s^2 - s - 2)L\{y\} + s - 2 - 1 = \frac{40}{s^2 + 4}$$

$$\Rightarrow L\{y\} = \frac{s-1}{s^2 - s - 2} + \frac{40}{(s^2 - s - 2)(s^2 + 4)}$$

$$\Rightarrow L\{y\} = \frac{-\lambda^2 - 3}{(\lambda-2)(\lambda+1)} + \frac{40}{(\lambda-2)(\lambda+1)(\lambda^2+4)}$$

$$\Rightarrow y(t) = L^{-1}\left[\frac{-1}{\lambda-2}\right] + L^{-1}\left[\frac{40}{(\lambda-2)(\lambda+1)(\lambda^2+4)}\right]$$

Now,  $L^{-1}\left[\frac{1}{\lambda-2}\right] = e^{2t}$  → (3)

Partial fractions

$$(4) \quad \frac{40}{(\lambda-2)(\lambda+1)(\lambda^2+4)} = \frac{A}{\lambda-2} + \frac{B}{\lambda+1} + \frac{C\lambda+D}{\lambda^2+4} \rightarrow (4)$$

$$\Rightarrow A = \frac{40}{(3)(8)} = \frac{5}{3}$$

$$B = \frac{40}{(-3)(5)} = -\frac{8}{3}$$

multiplying (4) with  $(\lambda-2)(\lambda+1)(\lambda^2+4)$

$$40 = (\lambda+1)(\lambda^2+4) \frac{5}{3} + (\lambda-2)(\lambda^2+4) \left(-\frac{8}{3}\right) + (C\lambda+D)(\lambda-2)(\lambda+1)$$

Equating coeffs of  $\lambda^3$  and constant

$$\Rightarrow 0 = \frac{5}{3} - \frac{8}{3} + C \Rightarrow C = 1$$

$$40 = \cancel{\frac{5}{3}} + \frac{20}{3} + \frac{64}{3} - 2D \Rightarrow D = -6$$

$$\therefore L^{-1}\left[\frac{40}{(\lambda-2)(\lambda+1)(\lambda^2+4)}\right] = L^{-1}\left[\frac{5}{3(\lambda-2)}\right] - \frac{8}{3}L^{-1}\left[\frac{1}{\lambda+1}\right] + L^{-1}\left[\frac{1}{\lambda^2+4}\right] - L^{-1}\left[\frac{6}{\lambda^2+4}\right]$$

$$= \frac{5}{3}e^{2t} - \frac{8}{3}e^{-t} + \cos 2t - 3 \sin 2t$$

Please check it

$$\therefore y(t) = -e^{2t} + \frac{5}{3}e^{2t} - \frac{8}{3}e^{-t} + \cos 2t - 3 \sin 2t$$

$$\Rightarrow y(t) = \frac{\frac{2}{3}e^{2t} - \frac{8}{3}e^{-t} + \cos 2t - 3 \sin 2t}{2e^{2t} - 4e^{-t}},$$

7. (a) A uniform chain of length  $l$  hangs between two points  $A$  and  $B$  which are at a horizontal distance  $a$  from one another, with  $B$  at a vertical distance  $b$  above  $A$ . Prove that the parameter of the catenary is given by

$$2c \sinh(a/2c) = \sqrt{l^2 - b^2}$$

Prove also that, if the tensions at  $A$  and  $B$  are  $T_1$  and  $T_2$  respectively,  $T_1 + T_2 = W \sqrt{1 + \frac{4c^2}{l^2 - b^2}}$  and  $T_2 - T_1 = \frac{Wb}{l}$ , where  $W$  is the weight of the chain. (18)

IMS-Institute Of Mathematical Sciences

$$P + P + K \frac{dP}{dn} = n^u, P \frac{dP}{dn} - 4n^3 P$$

$$2P + K \frac{dP}{dn} = 2n^3 P \frac{dP}{dn}$$

$$y + Pn = R^u P$$

$$\int_0^{nh} \frac{\log(nm)}{nm} \frac{nm}{nm} \frac{nm}{nm}$$

$$\frac{n^2 h^2 \log(h^2 + h^2)}{h^2 + h^2}$$

$$\lim_{h \rightarrow 0, k \rightarrow 0} \frac{f(x+h, y+k) - f(x, y)}{\sqrt{h^2 + k^2}}$$

$$(x+h)^2 + y^2 \log(n^2 + h^2) - (x^2 + y^2) \log(n^2 + h^2)$$

$$\lim_{h \rightarrow 0} \frac{h^2 \log h^2}{h} \frac{h^2 \log h}{h}$$

7. (b) A solid hemisphere rests on a plane inclined to the horizon at an angle  $\alpha < \sin^{-1} \frac{3}{8}$ , and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable. (14)

IMS-Institute Of Mathematical Sciences

IMS-Institute Of Mathematical Sciences



Head Office: 305-106, Top Floor, Maithrijeen Tower, Dr. Mahatma Nagar, Delhi-110009.  
Branch Office: 25/8, Chhajedpur Market, Delhi-110068  
Ph.: 911-45625587, 09999329111, 09999137823 || [www.ims-institute.com](http://www.ims-institute.com) || [www.ims4mathslearning.com](http://www.ims4mathslearning.com) || Email: [ims4ms2010@gmail.com](mailto:ims4ms2010@gmail.com)

P.T.O.

7. (c) A particle is projected vertically upwards with velocity  $u$ , in a medium where resistance is  $kv^2$  per unit mass for velocity  $v$  of the particle. Show that the greatest height attained by the particle is  $\frac{1}{2k} \log \frac{g+ku^2}{g}$  (18)

IMS-Institute Of Mathematical Sciences

8. (a) (i) If  $\phi(x, y, z) = xy^2z$  and  $\mathbf{f} = xzi - xyj + yz^2k$ , show that  $\frac{\partial^3}{\partial x^2 \partial z}(\phi \mathbf{f})$  at  $(2, -1, 1)$  is  $\mathbf{f} = 4i + 2j$ .  
(ii) Prove that  $\nabla \times (\mathbf{F} \times \mathbf{r}) = 2\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{r} + (\mathbf{r} \cdot \nabla)\mathbf{F}$  (12)

$$(i) \phi \vec{f} = (x^2y^2z^2) \hat{i} - (x^2y^3z) \hat{j} + (xy^3z^3) \hat{k}$$

$$\Rightarrow \frac{\partial}{\partial x} (\phi \vec{f}) = (2xy^2z^2) \hat{i} - (2xy^3z) \hat{j} + (y^3z^3) \hat{k}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} (\phi \vec{f}) = (2y^2z^2) \hat{i} - (2y^3z) \hat{j}$$

$$\Rightarrow \frac{\partial^3}{\partial x^2 \partial z} (\phi \vec{f}) = (4y^2z) \hat{i} - (2y^3) \hat{j}$$

$$\text{at } (2, -1, 1) = 4 \hat{i} + 2 \hat{j}$$

~~at  $(2, -1, 1)$~~

$$(iii) \quad \underline{\nabla \times (\vec{F} \times \vec{A})}$$

We know

$$\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B}) \vec{A} + (\vec{B} \cdot \nabla) \vec{A}$$

$$- (\vec{A} \cdot \nabla) \vec{B} - (\nabla \cdot \vec{A}) \vec{B}$$

$$\therefore \nabla \times (\vec{F} \times \vec{A}) = (\nabla \cdot \vec{A}) \vec{F} + (\vec{A} \cdot \nabla) \vec{F}$$

$$- (\nabla \cdot \vec{F}) \vec{A} - (\vec{F} \cdot \nabla) \vec{A}$$

$$= (\vec{A} \cdot \nabla) \vec{F} - (\nabla \cdot \vec{F}) \vec{A} + (\vec{A}) \vec{F} - (\vec{F} \cdot \nabla) \vec{A}$$

as  $\nabla \cdot \vec{A} = 3$

Now,  $(\vec{F} \cdot \nabla) \vec{A} = (F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}) (x \hat{i} + y \hat{j} + z \hat{k})$

$$= F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} = \vec{F}$$

$$\nabla \times (\vec{F} \times \vec{A}) = (\vec{A} \cdot \nabla) \vec{F} - (\nabla \cdot \vec{F}) \vec{A} + 2 \vec{F}$$

8. (b) A vector function  $f$  is the product of a scalar function and the gradient of a scalar function, show that  
 $f \cdot \operatorname{curl} f = 0$  (08)

Given  $\vec{f} = \phi \nabla \psi$

Now,  $\nabla \times \vec{f} = \nabla \times [\phi \nabla \psi]$

$$\begin{aligned} &= (\nabla \times \phi)(\nabla \psi) \\ &= (\nabla \phi) \times (\nabla \psi) + \phi [\nabla \times (\nabla \psi)] \\ &= (\nabla \phi) \times (\nabla \psi) \quad \left[ \because \operatorname{curl}(\operatorname{grad} \phi) = 0 \right] \end{aligned}$$

$\therefore \vec{f} \cdot \operatorname{curl} \vec{f}$

$$= (\phi \nabla \psi) \cdot [(\nabla \phi) \times (\nabla \psi)]$$

$$= \phi \left[ \nabla \psi \cdot [(\nabla \phi) \times (\nabla \psi)] \right]$$

$$= \phi \begin{bmatrix} \nabla \psi & \nabla \phi & \nabla \psi \end{bmatrix}$$

$\stackrel{\text{Ob}}{=} \phi \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = 0 \quad \left[ \because [\vec{a} \vec{b} \vec{c}] = 0 \text{ if two of the vectors are same} \right]$

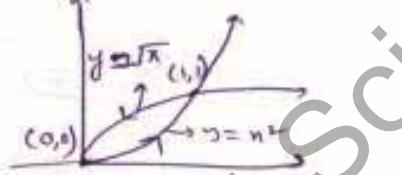
$\therefore \vec{f} \cdot \operatorname{curl} \vec{f} = 0$

8. (c) Verify Green's theorem in the plane for  $\int_C [(2xy - x^2)dx + (x^2 + y^2)dy]$ , where  $C$  is the boundary of the region enclosed by  $y = x^2$  and  $y^2 = x$  described in the positive sense. (12)

*Green's theorem*

$$\oint_C (M dx + N dy) = \iint_S \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

region



L.H.S

along  $y = x^2$

$$\begin{aligned} I_1 &= \int_{y=x^2} (2xy - x^2) dx + (x^2 + y^2) dy \\ &= \int_0^1 (2x^3 - x^2) dx + (x^2 + x^4) 2x dx \\ &= \int_0^1 (4x^3 - x^2 + x^5) dx = \left[ x^4 - \frac{x^3}{3} + \frac{x^6}{3} \right]_0^1 = \frac{1}{3} \end{aligned} \rightarrow ①$$

along  $y = \sqrt{x}$   $m = y^{-1}$

$$\begin{aligned} I_2 &= \int_0^1 (2y^3 - y^4) (2y dy) + (y^4 + y^2) dy \\ &= \int_0^1 (5y^4 - 2y^5 + y^2) dy = \left[ y^5 - \frac{2y^6}{3} + \frac{y^3}{3} \right]_0^1 = -\frac{1}{3} \end{aligned} \rightarrow ②$$

$$\oint_C (M dx + N dy) = I_1 + I_2 = 0 \rightarrow ③$$

$$\text{Now, } \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (\alpha x) - (\alpha y) \\ = 0$$

$$\therefore \iint_S \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = 0 \rightarrow \textcircled{4}$$

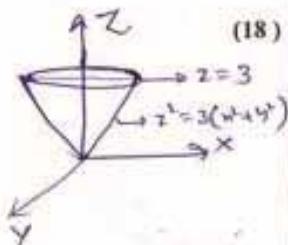
∴ from ③ + ④

Green's theorem verified.

8. (d) By using Gauss divergence theorem evaluate  $\iint_S (x^2 + y^2) dS$ , where  $S$  is the surface of the cone  $z^2 = 3(x^2 + y^2)$  bounded by  $z = 0$  and  $z = 3$ .

Gauss divergence theorem

$$\iint_S (\vec{F} \cdot \hat{n}) dS = \iiint_V (\nabla \cdot \vec{F}) dV$$



Given  $\vec{F} \cdot \hat{n} = (x^2 + y^2)$

But  $\hat{n} = \frac{6x\hat{i} + 6y\hat{j} - 2z\hat{k}}{\sqrt{36x^2 + 36y^2 + 4z^2}}$  from  $3x^2 + 3y^2 - z^2 = 0$   
(Surface)

$$= \frac{6x\hat{i} + 6y\hat{j} - 2z\hat{k}}{\sqrt{12z^2 + 4z^2}} = \frac{3x\hat{i} + 3y\hat{j} - z\hat{k}}{2z}$$

$$\text{If } \vec{F} \cdot \left( \frac{3x\hat{i} + 3y\hat{j} - z\hat{k}}{3z} \right) = x^2 + y^2$$

$$\Rightarrow \vec{F} = \cancel{\frac{3xz}{3}\hat{i}} + \frac{3yz}{3}\hat{j}$$

$$\text{Now, } \nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(\frac{3xz}{3}\right) + \frac{\partial}{\partial y}\left(\frac{3yz}{3}\right)$$

$$= \cancel{\frac{4z}{3}}$$

$$\therefore \iint_S (\vec{F} \cdot \hat{n}) dS = \iiint_V (\nabla \cdot \vec{F}) dV$$

$$\Rightarrow \iint_S (x^2 + y^2) dS = \iiint_V \left(\frac{4z}{3}\right) dx dy dz$$

$$= \iint_{\substack{z=0 \\ z=\sqrt{x^2+y^2}}} \left(\frac{4z}{3}\right) dx dy$$

$$= \frac{2}{3} \iint [z^2]_{\sqrt{x^2+y^2}}^3 dx dy$$

$$= \frac{2}{3} \iint (9 - 3x^2 - 3y^2) dx dy$$

$$(let) A = 2 \iint (3 - x^2 - y^2) dx dy$$

$$\text{on } \underline{x^2 + y^2 = 3}$$

$$\text{Let } x = r \cos \theta ; y = r \sin \theta$$

$$\Rightarrow dx dy = r dr d\theta$$

$$\begin{aligned}
 A &= 2 \int_0^{\sqrt{3}} \int_0^{2\pi} (3 - \lambda^2) \, \lambda \, d\lambda \, d\theta \\
 &= 2(2\pi) \int_0^{\sqrt{3}} (3\lambda - \lambda^3) \, d\lambda \cancel{d\theta} \\
 &= (4\pi) \left[ \frac{3\lambda^2}{2} - \frac{\lambda^4}{4} \right]_0^{\sqrt{3}} \\
 &= (4\pi) \left[ \frac{9}{2} - \frac{9}{4} \right]
 \end{aligned}$$

$$\underline{A = 9\pi}$$

$$\begin{aligned}
 \therefore \iint_S (x^2 + y^2) \, ds &= \underline{9\pi}
 \end{aligned}$$

## ROUGH SPACE

$$\begin{aligned}
 & \nabla(\vec{F} \times \vec{A}) = (\vec{A} \cdot \nabla) \vec{F} + (\vec{F} \cdot \nabla) \vec{A} - (\vec{F} \cdot \nabla) \vec{A} + (\nabla \cdot \vec{F}) \vec{A}, \quad \vec{F} \cdot \frac{\partial}{\partial n} \\
 & = 3 \vec{F} \\
 & \frac{(2x^2 - 4t - y)(-8y^3)}{(x^2 + y^2 + 1)^3} + \int \frac{(2y)}{x^3} \frac{1}{(x^2 + y^2 + 1)^2} d^n \\
 & \vec{A} = \left( \frac{6x^2 + 6y^2 - 2z^2}{\sqrt{36(x^2 + y^2) + 4z^2}}, \frac{-4x^2}{3(x^2 + y^2)}, \frac{-4z}{3(x^2 + y^2)} \right) = 3C(x^2 + y^2 + 1)^3 \\
 & \vec{F} \cdot \vec{A} - \int \vec{F} \cdot d\vec{S} = 0 \\
 & \int \vec{F} \cdot d\vec{S} = \int \int (\nabla \cdot \vec{A}) dS
 \end{aligned}$$

$$\begin{aligned}
 & \vec{F} = \frac{20}{(-D-B)} \sin 2t \\
 & \vec{A} = \frac{20(D-B)}{(D-B)^2 - 36} \sin 2t \\
 & \vec{A} = \frac{20(D-B)}{(D-B)^2 + 36} \sin 2t \\
 & T(\alpha M + \beta N) = \Lambda(\alpha M + \beta N) \\
 & = \alpha(T(M)) + \beta(T(N)) \\
 & \Delta \cdot \vec{A} = (2)
 \end{aligned}$$

**END OF THE EXAMINATION**

$$\frac{D_1 - D_1 + 1}{2} = \frac{1 \pm \sqrt{1 - 4}}{2}$$

$$D_1 = \frac{1 \pm i\sqrt{3}}{2} [n(2P) - n^2] = \frac{2P - 3n^2 P^2}{4n^2 + 4nP + 1}$$

$$a \left( \frac{dP}{dn} - 3nP^2 \right) = 2P \left( n \frac{dP}{dn} - 1 \right)$$

$$(D+1)(D^2 - 2\sqrt{P} + 1) = 0$$

$$\begin{aligned}
 & P\left(\frac{2 - 3n^2 p}{n(2np - 1)}\right) = \frac{\log(n \ln n)}{n} \text{ (using } D_1 = 1) \\
 & \frac{(A_1 n + B_1) + o(n^2)}{(n^2 + 4n + 2)} = \frac{n \log(n \ln n)}{n} - \frac{n \log(n \ln n)}{n} \\
 & \frac{(A_1 n + B_1) p(3n^2)}{\sqrt{2n^2 + 4n + 2}} = \frac{n \log(n \ln n)}{n} \\
 & \frac{(A_1 n + B_1) p(3n^2)}{\sqrt{2n^2 + 4n + 2}} = \frac{2n \log(n \ln n)}{2n} + \frac{(n^2 - 1)}{(n^2 + 4n + 2)} \frac{2n}{(2n)} \\
 & \frac{(A_1 n + B_1) p(3n^2)}{\sqrt{2n^2 + 4n + 2}} = \int_{1/2}^{1/2} \log(n \ln n) \, dn \quad \frac{\partial}{\partial n} = \frac{2n}{n^2 + 4n + 2}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Given } f(x) = \frac{\sin x}{x} \\
 & \text{Let } u = \sin x, v = \frac{1}{x} \\
 & \text{Then } du = \cos x dx, dv = -\frac{1}{x^2} dx \\
 & \text{By integration by parts, } \\
 & \int \frac{\sin x}{x} dx = \sin x \cdot \frac{1}{x} - \int \frac{1}{x} \cos x dx \\
 & = \frac{\sin x}{x} + \int \frac{\cos x}{x^2} dx \\
 & = \frac{\sin x}{x} + \frac{1}{x} \int \frac{\cos x}{x} dx \\
 & = \frac{\sin x}{x} + \frac{1}{x} \left[ \log(\tan x) \right]_0^{\pi/2} \\
 & = \frac{\sin x}{x} + \frac{1}{x} \log(\tan(\pi/2)) - \frac{1}{x} \log(\tan(0)) \\
 & = \frac{\sin x}{x} + \frac{1}{x} \log(\infty) - \frac{1}{x} \log(0) \\
 & = \frac{\sin x}{x} + \frac{1}{x} \cdot \infty \\
 & = \infty
 \end{aligned}$$

$$\nabla \left( -\frac{(\vec{a} \cdot \vec{n})}{\lambda^3} \right) = -T \delta(\text{arcos}\theta) + 4w \delta(\text{arcos}\theta) = 0$$

$$+ 2aT \sin\theta - 4wa \sin\theta) \delta\theta = 0$$

$$= \frac{\log(\text{cosec} - \cot^n)}{\text{cosec} + \cot^n} \quad \cancel{\frac{\text{cosec} - \cot^n}{\text{cosec} + \cot^n}}$$

$$2T = 4w$$

$$T = \frac{1}{2}(4w)$$

# OUR ACHIEVEMENTS

from 2008 to 2013

**IMS™**  
(INSTITUTE OF MATHEMATICAL SCIENCES)



**HIMANSHU GUPTA**

Total Marks in Maths (Opt.)

**430 / 600**



**ARITH JOHN JOSHUA**  
**AIR-78**  
(IAS-2013)



**SUMIT KUMAR**  
**AIR-81**  
(IAS-2013)



**B. SASHI KANT**  
**AIR-111**  
(IAS-2013)



**GOWTHAM POTRU**  
**AIR-318**  
(IAS-2013)



**RAVINDER SINGH**  
**AIR-333**  
(IAS-2013)



**ASHISH MODI**  
**AIR-350**  
(IAS-2013)



**PARAS MANI TRIPATHI**  
**AIR-391**  
(IAS-2013)



**NIRAMIL GOYAL**  
**AIR-399**  
(IAS-2013)



**NITISH K.**  
**AIR-547**  
(IAS-2013)



**KULWAJ SINGH**  
**AIR-552**  
(IAS-2013)



**VALLURU KRANTHI**  
**AIR-562**  
(IAS-2013)



**SANTOSH KUMAR**  
**AIR-1013**  
(IAS-2013)



**RAMESH RAJAWAT**  
**AIR-76**  
(IAS-2012)



**ANKIT VERMA**  
**AIR-247**  
(IAS-2012)



**B. SASHI KANT**  
**AIR-329**  
(IAS-2012)



**KRISHNA KANT**  
**AIR-550**  
(IAS-2012)



**YOGESH GANG**  
**AIR-560**  
(IAS-2012)



**PRADEEP SINGH**  
**AIR-633**  
(IAS-2012)



**KUTUB BARBARI**  
**AIR-655**  
(IAS-2012)



**SANJAY K. JAIN**  
**AIR-667**  
(IAS-2012)



**SANTOSH KUMAR**  
**AIR-849**  
(IAS-2012)



**MEET KUMAR**  
**AIR-944**  
(IAS-2012)



**AMIT MUKHERJEE**  
**AIR-25**  
(IAS-2011)



**AYAY KIRAN TOLLA**  
**AIR-88**  
(IAS-2011)



**AWAKASH KUMAR**  
**AIR-168**  
(IAS-2011)



**BULNEET SINGH**  
**AIR-220**  
(IAS-2011)



**AJIT PRASAD SINGH**  
**AIR-288**  
(IAS-2011)



**ART TADAV**  
**AIR-372**  
(IAS-2011)



**RAVI VERMA**  
**AIR-485**  
(IAS-2011)



**NEONKA AGARWAL**  
**AIR-538**  
(IAS-2011)



**B.L. KUMAR**  
**AIR-796**  
(IAS-2011)



**KANCHAN P. DIXIT**  
**AIR-223**  
(IAS-2011)



**SHASHIKANT KUMAR**  
**AIR-154**  
(IAS-2010)



**ANIL KUMAR**  
**AIR-276**  
(IAS-2010)



**RAHESH AGRAWAL**  
**AIR-362**  
(IAS-2009)



**AJIT PRASAD SINGH**  
**AIR-497**  
(IAS-2009)



**MAHENDRA KUMAR**  
**AIR-47**  
(IAS-2009)



**A. ARUN**  
**AIR-140**  
(IAS-2009)



**NISHA GUPTA**  
**AIR-507**  
(IAS-2008)



**K.V.S.R. KISHORE**  
**AIR-575**  
(IAS-2008)



**SIDDHARTH CHAUHAN**  
**AIR-16**  
(IIT-2013)



**RISHABH KUMAR**  
**AIR-29**  
(IIT-2013)



**RAHUL KUMAR**  
**AIR-39**  
(IIT-2013)



**ANUPAM KUMAR**  
**AIR-72**  
(IIT-2013)



**ABHILASH KUMAR**  
**AIR-7**  
(IIT-2012)



**DEEPAK KUMAR**  
**AIR-32**  
(IIT-2012)



**RAHUL KUMAR**  
**AIR-48**  
(IIT-2012)



**RAHUL KUMAR**  
**AIR-72**  
(IIT-2012)



**RAHUL KUMAR**  
**AIR-5**  
(IIT-2011)



**RAHUL KUMAR**  
**AIR-11**  
(IIT-2011)



**RAHUL KUMAR**  
**AIR-4**  
(IIT-2010)



**RAHUL KUMAR**  
**AIR-36**  
(IIT-2010)



**RAHUL KUMAR**  
**AIR-80**  
(IIT-2009)



**RAHUL KUMAR**  
**AIR-23**  
(IIT-2009)



**RAHUL KUMAR**  
**UP-PCS**  
(2011)